LECTURE I

X closed Riemann Surface (RS) of genus $g \geq 2$ 

$p : \tilde{X} \to X$ universal covering, $\pi_1 X \curvearrowright \tilde{X}$, $\gamma \cdot x = \gamma(x)$, holomorphic covering transformations

__Uniformization Theorem__ $\exists$ holomorphic diffeo. $D : \tilde{X} \to \mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.

Set $\rho(\gamma) = D\gamma D^{-1} : \mathbb{H} \to \mathbb{H}$ $\Rightarrow$ $\rho : \pi_1 X \to \operatorname{Aut}(\mathbb{H})$

$$\operatorname{Aut}(\mathbb{H}) = \{T : \mathbb{H} \to \mathbb{H} \mid T(z) = \frac{az+b}{cz+d},\ ad-bc=1,\ a,b,c,d \in \mathbb{R}\}$$

Note: $1 \to \{\pm I\} \to SL_2\mathbb{R} \to \operatorname{Aut}(\mathbb{H}) \to 1$ $\Longrightarrow$ $\operatorname{Aut}(\mathbb{H}) = PSL_2\mathbb{R}$.

$\therefore$ $\rho(\pi_1 X) = G < PSL_2\mathbb{R}$, write $p : \mathbb{H} \to X = G \backslash \mathbb{H}$.

__Def'n__ The hyperbolic metric on $\mathbb{H}$ is $ds_{\mathbb{H}} = \frac{|dz|}{\operatorname{Im} z}$

distance function $d(z,w) = \inf\limits_{\substack{\gamma : [0,1] \to \mathbb{H} \\ \gamma(0) = z \\ \gamma(1) = w}} l_{\mathbb{H}}(\gamma) \left(= \int_0^1 \frac{|\gamma'(t)|\,dt}{\operatorname{Im}(\gamma(t))} \right)$

__Proposition__ ① $PSL_2\mathbb{R} = \operatorname{Isom}^+(\mathbb{H}, ds_{\mathbb{H}})$, ② geodesics in $\mathbb{H}$ are arcs of lines & circles $\perp$ to $\mathbb{R}$. pf exercise. $\square$

__Proposition__ (Classification of Isometries) Given $T \in PSL_2\mathbb{R} \setminus \{id\}$ then

ⓐ T fixes a pt in $\mathbb{H}$ and is conjugate to $T_0(z) = \frac{\cos\theta\, z + \sin\theta}{-\sin\theta\, z + \cos\theta}$, (__elliptic__)

ⓑ T acts by translation along a unique geodesic axis and is conjugate to $T_0(z) = \lambda^2 z = \frac{\lambda z + 0}{0z + \lambda^{-1}}$, $\lambda > 0$. (__hyperbolic__) __OR__

ⓒ T fixes a unique pt on $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, $\exists \{z_n\} \subset \mathbb{H}$ w/ $d_{\mathbb{H}}(z_n, T(z_n)) \to 0$ and T is conjugate to $T_0(z) = z+1 = \frac{1z+1}{0z+1}$ (__parabolic__) / pf exercise $\square$

<u>Observe</u>: can define a metric on X so $p: \mathbb{H} \to X$ is a local isometry — why? length of path is length of lift (indep of lift...)

<u>Killing-Hopf Theorem</u> Any metric locally isometric to $(\mathbb{H}, ds^2_{\mathbb{H}})$ has universal cover isometric to $(\mathbb{H}, ds_{\mathbb{H}})$.

<u>Proposition</u> If $\gamma: S^1 \to X$ is a non-null homotopic closed curve, and $[\gamma_0] \in \pi_1 X$ a representative of conjugacy class defined by γ, then:

(i) γ is homotopic to a "unique" closed geodesic $\gamma^*: S^1 \to X$

(ii) $\rho(\gamma_0)$ is hyperbolic, and

(iii) $\text{length}(\gamma^*) = 2\cosh^{-1}\left(\frac{|\text{tr}\,\rho(\gamma_0)|}{2}\right)$

Pf: exercise □

<u>Proposition</u> Suppose $f: X \to Y$ is an o.p. homeo (X, Y RS's). Then f is holomorphic iff f is and isometry (for hyp. metrics)

pf lift:
$$\begin{tikzcd} \mathbb{H} \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{H} \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
$\tilde{f}$ is hol. iff in $PSL_2\mathbb{R}$ iff isometry □

Using hyperbolic geometry, we can construct <u>all</u> genus $g \geq 2$ RS (up to isometry / holomorphic diffeo). — 4 <u>steps</u>

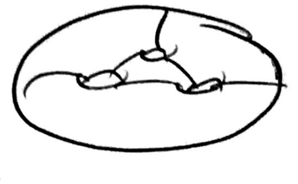
<u>Step 1</u> Proposition Right-angled hyperbolic hexagons are parametrized by alternating side lengths. pf exercise:  □

Gluing together isometric R-A hexagons $\Rightarrow$ hyperbolic pants w/ geod. boundary

<u>Step 2</u> <u>Proposition</u> Hyperbolic pants w/ geodesic boundary are parameterized by boundary lengths

pf exercise (find hexagons) $\square$

<u>Step 3</u> Build surfaces by gluing pants



- length must match
- flexibility in gluing: ~~twist~~ twisting.

<u>Step 4</u> <u>Proposition</u> Any genus $g \geq 2$ hyp. surface is isometric to one from Step 3.

pf exercise (pick top pants, straighten to geodesics, prove still pants decomposition).

Teichmüller Space

S a topological surface of genus $g \geq 2$. ~~A RS struct~~

A RS-structure on S (is "RS clothes") is an o.p. homeomorphism $f: S \to X$ to a RS X. Say:

$$f: S \to X \sim h: S \to Y$$

if $\exists$ hol. diffeo $\sigma: X \to Y$ so $\sigma \circ f \simeq h$:

$$\begin{tikzcd} & X \arrow[d, "\sigma"] \\ S \arrow[ur, "f"] \arrow[dr, "h"'] \arrow[r, phantom, "\simeq"] & {} \\ & Y \end{tikzcd}$$

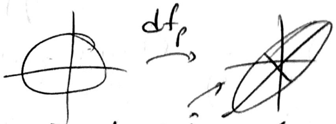
Write $[X, f]$ for equiv. class.

<u>Defn</u> $T(S) = \{ [X, f] \mid f: S \to X \text{ is a RS structure} \}$

This defines $T(S)$ as a <u>set</u>. We give it a metric, but this requires some definitions ...

$U, V \subset \mathbb{C}$ open, $f : U \to V$ o.p. <u>diffeo</u> is K-quasi conformal if $\forall p \in U$

$$(*) \quad \frac{\|df_p\|^2}{\det(df_p)} \leq K \iff \text{ratio } \frac{\text{maj}}{\text{min}} \leq K$$



<u>Alt:</u> $f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$, $f_{\bar z} = \frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$

<u>CR equis</u>: $f_{\bar z} = 0$ in this case $f' = f_z$.

$\mu_f(z) = \frac{f_{\bar z}}{f_z}$ measures failure of being holomorphic

$(*)$ is equivalent to ($\|\mu_f\|_\infty < 1$ and)

$$\frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|} \leq K$$

Set $K_f = \sup_z \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}$

<u>Defn</u> An o.p. <u>homeo</u> is K-q.conformal if it is a limit (locally uniform) of K-q. conformal diffeos

[[Correction: limit of $(K+\varepsilon)$-q.conf. diffeos $\forall \varepsilon > 0$]]

Extend to $f : X \to Y$ in local coords.

Proposition $X \xrightarrow{f} Y \xrightarrow{h} Z$ f K-q.conf. h K'-q.conf.

then: • f^{-1} is K-q.conf

• hf is KK'-q.conf.

pf exercise (compute for diffeos, take a limit).

Theorem K-q.conf maps $\{f : X \to Y\}$ is compact. (X, Y closed RS). $\square$

Theorem (Weyl's Lemma) $f : X \to Y$ is 1-q.conf iff f is holomorphic. $\square$

Defn (Teich metric)

$$d_T([X,f],[Y,h]) = \inf_{\substack{F \simeq hf^{-1} : X \to Y \\ \text{q.conf}}} \tfrac{1}{2} \log K_F$$

Proposition d_T is a metric.

pf exercise (apply 3 facts above). $\square$