LECTURE II

An o.p. homeo $f: X \to Y$ (RS) is K-q.conf. if in loc. coords it has weak derivatives represented by locally integrable functions and $|f_{\bar z}| \le k|f_z|$ where $k = \frac{K-1}{K+1}$ [[precise defin]]

$K_f = \inf\{K \mid f \text{ is } K\text{-q.conf}\}$

Facts:

① f K-q.conf $\Rightarrow$ f is diff'ble a.e. and $\frac{\|df\|^2}{\det df} \le K$ a.e.

② f a diffeo at all but finite # of pts and if ⁅ holds a.e., then it is K-qconformal.

~~Proposition~~ Theorem (Grötzsch) For $b \ge a$, and

$R_a = [0,a]\times[0,1]$, $R_b = [0,b]\times[0,1]$, then

$f: R_a \to R_b$ K-q.conf, $f(\text{corners}) = \text{corners}$ & $f(0,0)=(0,0)$, then

$K \ge b/a$ w/ equality iff $f(x+iy) = (\frac{b}{a}x + iy)$.

pf: $|f_x(p)| \le |df_p| \le \sqrt{K}\sqrt{\det df_p}$ for a.e. p. So

$$\underbrace{b^2 \le \left(\int_0^1\int_0^a |f_x|\, du\, dv\right)^2}_{b \le \int_0^a |f_x|\, du} \le \left(\int_{R_a}|f_x|\, dA\right)^2 \le \int_{R_a} K\, dA \int_{R_a} \det df_p\, dA$$

$$= (Ka)b \Rightarrow \frac{b}{a}\le K \quad \square$$

Theorem $\mathrm{Mod}(S) = \mathrm{Homeo}^+(S)/_{\text{htpy}} \curvearrowright T(S)$ by $[f]\cdot[X,h] = [X, hf^{-1}]$ acts by isometries.

pf $d_T([X, hf^{-1}],[Y, gf^{-1}]) = \inf\limits_{F \simeq (\ldots)^{-1} \simeq (hf^{-1})(gf^{-1})^{-1} = hg^{-1}} \frac{1}{2}\log K_F = d_T([X,h],[Y,g])$

$\square$

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Connection to hyp. geom: $\forall$ closed curves $\alpha : S^1 \to S$

~~Wolpert~~ set $\ell_\alpha : T(S) \to \mathbb{R}$ defined by

$$\ell_\alpha([X, f]) = \text{length}\,((f\alpha)^*)$$

$*$ X-geod. htpc to $f\alpha$.

exercise well-defined.

Wolpert's Lemma $\forall [X,h], [Y,g] \in T(S)$, closed curve α,

$$\ell_\alpha([X,h]) \leq e^{2d_T([X,h],[Y,g])}\, \ell_\alpha([Y,g])$$

pf exercise (use modulus and appropriate covers as discussed in prob. session). $\square$

Corollary $\log(\ell_\alpha)$ is 2-lipschitz. $\square$

Teichmüller geodesics:

X R.S. (genus $g \geq 2$) $Q(X)$ = hol. quadratic diffls $\overset{\text{Riemann-Roch}}{\cong} \mathbb{C}^{3g-3}$

$\phi \in Q(X) \Rightarrow \phi = \phi_0(z)dz^2$ in local coords z.

$\phi \neq 0$ preferred coords: $\zeta : U \to \mathbb{C}$, $\zeta(p) = \int_{p_0}^{p} \sqrt{\phi} \Rightarrow \phi = d\zeta^2$ transition: $z \mapsto \pm z + c$

$U = (p_0, p_0)$

$\phi \longleftrightarrow \{\zeta_\alpha : U_\alpha \to \mathbb{C}\}$ loc. coords away from zeros
Eucl. metric away from zeros, completion adds cone pts at zero p w/ angle $\pi(d+2)$, d = degree of zero.

this data determines RS. str. & ϕ.

also get well-defined vertical/horizontal singular foliations.

((anything in $\mathbb{C}$ invariant by $z \mapsto \pm z + c$ can be transported to X))

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Get an action of $SL_2\mathbb{R}$ on all quadratic differentials on all RS ...

$A \in SL_2\mathbb{R}$, $\phi = \{ \zeta_\alpha : U_\alpha \to \mathbb{C} \}$ define

$A \cdot \phi = A \cdot \{ \zeta_\alpha : U_\alpha \to \mathbb{C} \} = \{ A \circ \zeta_\alpha : U_\alpha \to \mathbb{C} \}$

((here $A : \mathbb{C} \to \mathbb{C}$ is acting as $\mathbb{R}$-linear trans))

check: $\zeta_\beta \circ \zeta_\alpha^{-1}(z) = \pm z + c \Rightarrow (A \circ \zeta_\beta) \circ (A \circ \zeta_\alpha)^{-1}(z) = \pm z + c'$.

Special case $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$:

$$X, \phi \in Q(X), t \in \mathbb{R} \rightsquigarrow g_t \cdot (X, \phi) = (X_t^\phi, \phi_t)$$

note $X = X_t^\phi$ <u>as topological surfaces</u>.

write $g_t^\phi : X \to X_t^\phi$ for identity — ~~2~~ new RS str. on X.

<u>Teichmüller's Uniqueness Theorem</u>

$\forall$ $X, \phi \in Q(X)$, $t \in \mathbb{R}$ and $f : X \to X_t^\phi$ K-q. conf. w/ $f \simeq g_t^\phi$,

$K \geq e^{|2t|}$ w/ equality iff $f = g_t^\phi$.

[[generalization of Grötzsch's Thm]]

<u>Corollary</u> $\forall$ $X, \phi \in Q(X)$ the map $t \mapsto [X_t^\phi, g_t^\phi] \in T(X)$

is a geodesic.

<u>pf</u>: exercise (g_t^ϕ is optimal ~~q.c.~~ q conf. map) $\square$

((If $h : S \to X$ is a RS structure on S, get <u>isometry</u> $T(X) \to T(S)$ by ~~[...]~~ $[Y, g] \mapsto [Y, g \circ h]$ ... So, get geod. in $T(S)$))

This allows us to connect any 2 pts by a geodesic, but w/ more work... L^1-Norm on $Q(X)$, $\int_X |\phi| =$ Area flat metric $= \|\phi\|$

Theorem (Teichmüller, Ahlfors, Bers): X RS, ~~[illegible]~~,

the map
$$Q(X) \to T(X)$$
$$\phi \mapsto [X^{\phi}_{\|\phi\|}, g^{\phi}_{\|\phi\|}]$$
defines a homeomorphism.

note: $\mathbb{R}$lines through 0 in $Q(X)$ map isometrically to geodesics. $\square$

Difficulty in this theorem is variation of $g^{\phi}_{\|\phi\|}$ as ϕ varies, and proving continuous.

Exercise any 2 pts are in fact connected by a unique geodesic. Hint: use both theorems.

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