pseudo-Anosov

\[ f: S \to S \text{ an o.p. homeomorphism is called pseudo-Anosov (pA).} \]

If \( \exists \lambda = \lambda(f) > 1 \), a pA structure \( h: S \to X \) and \( \Phi \in \mathcal{A}(X) \) s.t. \( f \) is given in local preferred coordinate by

\[ x + iy \mapsto \lambda x + i \frac{1}{\lambda} y. \]

More precisely:

\[ S \xrightarrow{h} X \xrightarrow{\Phi} C \]

For \( \Phi \in \mathcal{A}(X) \), let \( g \) be a fixed point of \( \Phi \). Then \( \phi(\Phi) \) is a fixed point of \( \Phi \).

**Proposition**: If \( f: S \to S \) is a pA, then \( [f] \in T(S) \).

**Proof**: Exercise (unwound above in definition of equivalence for \( T(S) \)).

For \( [f] \in \text{Mod}(S) \), define

\[ T([f]) = \inf \left\{ \lambda(f): [f] \leq [g], [f] \neq [g] \right\} \]

if \( \inf \) is a min, we say \( T \) is **realized**.

We are now ready for...
Classification Theorem (Thurston, Nielsen, Benj.)

Given \([f] \in \text{Mod}(S)\), \(f_0 : S \to S\) a rep of \([f]\) st:

1. If \(\text{Tr}[f] > 0\) and is realized, then \(f_0\) is \(pA\) and \(\text{Tr}[f] = \text{sgn}(\lambda_1(h))\)
2. If \(\text{Tr}[f] = 0\), \(f_0\) has finite order \(n \geq 0\); so \(f_0^n = \text{id}_S\).
3. If \(\text{Tr}[f] < 0\) not realized, then \(\exists C \subset S\) a multicurve \(d(f_0(C)) < C\).

Further, \(\text{Tr}[f] = 0\) in last case \(\iff f_0\) has a multi-twist power...

Consider the theorem to classify \(S\) isometries \((H, \text{Id}_S)\).

pts of \(\mathcal{O}\) define...

Try to realize \(\text{Tr}[f]/[\Sigma_{1,n}, h_n] \neq 0\).

\(d_1([\Sigma_{1,n}, h_n], [f^j]) \rightarrow \text{Tr}[f]\).... find multicurve \(C \subset S\) w/\(\lambda_C < h_n\).

Then two facts: 1. Collar lemma; short curves have big collars

\(\frac{e}{\log(\lambda_C)} \leq \frac{r_C}{2}\)

(ii) Wirtinger Lemma

\[ f_{x_h}^{[\Sigma_{1,n}, h_n]} \leq e^{\frac{r_C}{2}} \text{ for } 0 \leq \log(\lambda_C) \leq 100 \]

\(\Rightarrow \lambda_C \text{ big, } f_{x_h} \text{ all } \lambda_C \text{ zero in } \mathbb{R}^+\)

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\(\Rightarrow \text{ no more than } 2^3 \text{ so repeat,} \)

\(\Rightarrow \text{ isn't multicurve }\)

PA stretch factors... Note by above, \(X(f) = X([f]) = \text{stretch factor}\) for \(f \neq 1\). Also follows from:

Theorem (Thurston) for any \(\alpha \in \pi_1(S)\), closed curve \(x\), any min norm on \(S\),

\[
\lim_{n \to \infty} \text{length}(f^n(x)) = \lambda \quad (f^n(x)) = \min_{m > 0} \max_{k \text{ geod. arc}} \text{dist}(f^n(x), f^m(x)).
\]

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Let $\Phi$ be a map of $\phi$-diffeomorphism from $\phi$. Note that $d$-lengths and $m$-lengths are comparable, up to a multiplicative constant $K(m,d)$, but $n$th root kills $K(m,d)$.

Theorem (Thurston) $f: S \to S$ a Markov Partition for $f$ and associated P-F matrix $A$ has $\lambda(f)$ as P-F eigenvalue $A = \lambda(f)$.

What does this mean? A Markov Partition: rectangle $R_1 \times \ldots \times R_k$ with $d$-horizontal/vertical sides (not $m$-diffeomorphic).

So that:

$A_{ij} = \#\text{ times } f(R_j) \text{ map to } R_i$

- Each metric determines widths $r_i$ for each $R_i$, $i = 1, \ldots, k$, Note:

$A W_j = \sum_{i=1}^{k} W_i A_{ij} \Rightarrow \lambda$ is eigenvalue w.r.t. metric $W$-

Cor: $A$ is an algebraic integer.

(Also comes from action of $H$, after passing to (holonomy trivializing cover) of $\phi$-manifold)
Theorem (Penner) \( \forall g \geq 2, \forall pA \ f: S_g \to S_g \) (see \( g \))

\[
\log(\lambda(f)) \geq \frac{\log(2)}{12g-12}
\]

and \( \exists pA \ f: S_g \to S_g \) s.t. \( \log(\lambda(f)) \leq \frac{\log(4)}{g} \)

For lower bound comes from Markov partition (uri))

\[ Z_{\gamma} \text{ P-F matrix, } \implies \exists \text{ Markov } \gamma \text{ s.t. } \gamma \approx g \text{ red's...} \]

Examples \( f : S_g \) 

... describe construction, but first, define:

\[
L_g = \min \{ \log(\lambda(f)) | f: S_g \to S_g \text{ pA} \}
\]

(Penner) \( \Rightarrow L_g \leq \frac{\log(4)}{g} \)

Improved to:

\[
L_g \leq \frac{2\log(1+\frac{\sqrt{5}}{2})}{g-1}
\]

(Heusener, Ham-Takacsawen, Asker, Dunfield)

Question (Penner) What is \( L_g \)?

Only known for \( g = 2 \) (Ham-Song, Cho-Han). May other related
special cases (Ko-Lee-Song, Ham-Song, Lanneau-Thiffeault, Boissy-Lanneau,
Lecchi-Singh).

Question (McMullen) Does \( \sum_{g=2}^{\infty} L_g \) converge?

To understand/approach this, need to understand where

\[
\uparrow \log(\lambda(f)) \leq \text{ constant}
\]

Unifying construction due to McMullen.
Let $f: S \to S$ be any $pA$, $M_f = \text{mapping torus}$.
$\psi$ suspension fibre on $M$,
locally $\psi(x,t) = (x, t+h_t)$.

1st return map of $\psi$ to $S = S \times \mathbb{S}^1 = S \times \mathbb{R}$ if $f$.

Theorem (Thurston-Fried). \ $f: S \to S \times \mathbb{R}$, $M = M_f$, then $E$:

1. open cone $C \subset H_2(M, \mathbb{R})$ containing $[S]$ and
   $h_\alpha \in C_2 := H_2(M, \mathbb{R})^\perp$ representing surface $S$.
   (so $[S_\alpha] = \alpha$) $\cup S_\alpha$ are and 1st return map a $pA$, $f_\alpha: S \to S$.

2. $\overline{X}: C \to \mathbb{R}$ linear function st.

3. $h: C \to \mathbb{R}$ continuous, homogeneous of degree -1 function st.
   $h(\alpha) = \log(\lambda f_\alpha)$. 

Observe: Let $S_{n \alpha} \subset C_2$ be a sequence of $\mathbb{S}^7$.

1. $\overline{X}(S_{n \alpha}) \to \overline{X}(S_{\alpha})$.
   $\overline{X}(S_{n \alpha}) = [S_{n \alpha}]$ so $\overline{X}(S_{\alpha}) = \overline{X}(S_{n \alpha})$.
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2. $h(\alpha, \overline{X}(S_{\alpha})) = \log(\lambda f_\alpha) |\overline{X}(S_{\alpha})| = \log(\lambda f_{\alpha n}) |\overline{X}(S_{\alpha})|.$

\begin{align*}
\frac{1}{\lambda f_{\alpha n}} |\overline{X}(S_{\alpha})| &= \log(\lambda f_{\alpha n}) |\overline{X}(S_{\alpha})| \to \log(\lambda f_\alpha) |\overline{X}(S_{\alpha})| = \text{constant}.
\end{align*}

Theorem (Furber-L-Hughes) Any sequence $(f_n: S \to S)_{n \geq 1}$ w/ $\log(\lambda f_{n \alpha}) \leq C$

comes from above construction, up to subsequence as bouncing off cone $pA$. 