INSTITUT FOURIER
Laboratoire de Mathématiques
UMR 5582 CNRS-UGA



# LECTURE III

## pseudo-Anosovs

$f : S \to S$ an o.p. homeomorphism is called pseudo-Anosov (pA)

if $\exists\, \lambda = \lambda(f) > 1$, a RS structure $h : S \to X$, and

$\phi \in Q(X)$ s.t. $f$ is given in local preferred coords by

$$x + iy \longmapsto \lambda x + i \tfrac{1}{\lambda} y .$$

More precisely:

$$\begin{tikzcd} S \arrow[r, "h"] \arrow[d, "f"'] & X \supset U_\alpha \arrow[r] \arrow[d] & \mathbb{C} \arrow[d, "\substack{x+iy \\ \downarrow \\ \lambda x + i\frac{1}{\lambda} y}"'] \\ S \arrow[r] & X \supset U_\beta \arrow[r] & \mathbb{C} \end{tikzcd}$$

Note:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "g_t"] \arrow[d, "\substack{x+iy \\ \downarrow \\ \lambda x + i\frac{1}{\lambda} y}"] & \mathbb{C} \arrow[d, "\substack{x+iy \\ \downarrow \\ \lambda x + i\frac{1}{\lambda} y}"] \\ \mathbb{C} \arrow[r, "g_t"'] & \mathbb{C} \end{tikzcd}$$

So we can use $g_t^\phi h : S \to X_t^\phi$ and q.d.ff. $\phi_t = g_t \phi \in Q(X_t^\phi)$ and get same local picture. $\therefore$

**Proposition** $f : S \to S$ pA, then $[f] \curvearrowright T(S)$ by translation along a geodesic $Ax([f])$ w/ translation length $\log(\lambda(f))$

pf: Exercise (unravel above w/ defn of equivalence for $T(S)$) $\square$.

---

For $[f] \in Mod(S)$, define

$$\tau([f]) = \inf_{[X,h] \in T(S)} d_T\left([X,h], [f] \circ [X,h]\right)$$

if inf. is a min, we say $\tau$ is <u>realized</u>.

We are now ready for ...

# Classification Theorem (Thurston, Nielsen, Bers).

Given $[f] \in \mathrm{Mod}(S)$, $\exists\, f_0 : S \to S$ a rep'n of $[f]$ s.t.:

① If $\tau[f] > 0$ and is realized, then $f_0$ is pA and $\tau[f] = \log(\lambda(f_0))$

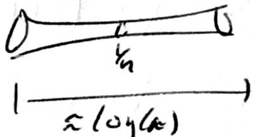
② If $\tau[f] = 0$ - - - - - - - - $f_0$ has finite order $n \geq 0$; so $f_0^n = id_S$.

③ If $\tau[f] \geq 0$ -- not realized, then $\exists\, C \subset S$ a multicurve s.t. $f_0(C) = C$

[Furthermore, $\tau[f] = 0$ in last case $\Leftrightarrow$ $f_0$ has a multitwist power...
and $\tau[f] > 0$ in last case $\Leftrightarrow$ $f_0$ has a "pA component".]

(compare this Thm to class. of isometries of $(\mathbb{H}, d_{\mathbb{H}})$)

pfs of ① & ② define... ③ try to realize $\tau[f]$ $\exists\, \{[X_n, h_n]\}$ s.t. $d_T([X_n,h_n], [f]\cdot[X_n,h_n]) \to \tau[f]$, ... Find curves $\alpha_n \subset S$ w/ $\ell_{\alpha_n}([X_n,h_n]) < \frac{1}{n}$. Then two facts: (i) Collar lemma: short curves have big collars



, and

(ii) Wolpert's Lemma

$\Longrightarrow \ell_{f^k\alpha_n}[X_n,h_n] \leq 1/100$ for $0 \leq k \leq \frac{\log n - 100}{2}$

$\Longrightarrow \alpha_n, f\alpha_n, f^2\alpha_n, \ldots, f^k\alpha_n$ all zero int #.

$\Longrightarrow$ n suff. large, more than $3g-3$, so repeat,

$\Longrightarrow$ inv't multicurve. $\square$

---

pA stretch factors. Note by above, $\lambda(f) = \lambda([f])$ = stretch factor (or dilatation) for $f$ pA. Also follows from:

Theorem (Thurston) for any ess. closed curve $\alpha$, any metric $m$ on $S$, $\lim_{n\to\infty} \sqrt[n]{\mathrm{length}_m((f^n\alpha)^*)} = \lambda$ $\quad (f^n\alpha)^* = m$-min. geod. htpc. to $f^n\alpha$.

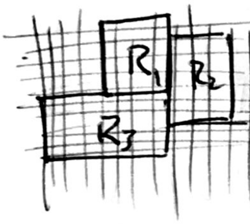
pf exercise: (~~obvious~~ "obvious" for q-diff metric from $\phi$, note that $\phi$-lengths and m-lengths are comparable, up to mult. constant $K(m,\phi)$, but $n^{th}$ root kills $K(m,\phi)$ ) $\square$.

---

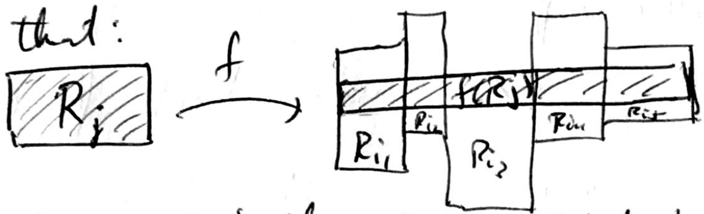
**Theorem** (Thurston) $f: S \to S$ pA, then $\exists$ a Markov Partition for $f$ and associated $\mathbb{Z}_{\geq 0}$ P-F matrix $A$ has ← Perron-Frobenius.

P-F eigenvalue $\lambda = \lambda(f)$.

What does this mean?: Markov Partition: rectangles $R_1, \ldots, R_k$ w/ horizontal/vertical sides (w.r.t. q-diff'l $\phi$).

$\phi$ 


so that:




(similar pic for $f^{-1}$, but w/ vert. sides.)

$$A_{ij} = \#\text{ times } f(R_j) \text{ maps over } R_i$$

$\phi$-Euc. metric determines widths $W_i$ for each $R_i$, $i = 1, \ldots, k$. Note:

$$\lambda W_j = \sum_{i=1}^{k} W_i A_{ij} \implies \lambda \text{ is eigenvalue w/ eigenvector } W = (w_1, \ldots, w_k)$$

**Cor** $\lambda$ is an alg. integer.

(( ~~this~~ also comes from action on $H^1$, after passing to holonomy trivializing branched cover of $\phi$-metric ))

INSTITUT FOURIER
Laboratoire de Mathématiques
UMR 5582 CNRS-UGA


Theorem (Penner) $\forall g \geq 2$, $\forall$ pA $f: S_g \to S_g$ (genus $g$ surface)

$$\log(\lambda(f)) \geq \frac{\log(2)}{12g-12}$$

and $\exists$ pA $f_g: S_g \to S_g$ st $\log(\lambda(f_g)) \leq \frac{\log(11)}{g}$

pf lower bound comes from Markov partition (basically)
— $\mathbb{Z}_{\geq 0}$ P-F matrix, $\exists$ Mark. Part. w/ $\leq 9g-9$ rect's...

Examples $f_g$? ... describe construction, but first, define:

$$L_g = \min\{\log(\lambda(f)) \mid f: S_g \to S_g \text{ pA}\}$$

(Penner) $\Rightarrow$ $L_g \leq \frac{\log(11)}{g}$.

Improved to:
$$L_g \leq \frac{\cancel{2}\log(\frac{1+\sqrt{5}}{2})}{g-1}$$ (Hironaka, Kin-Takasawa, Aaber-Dunfield)

Question (Penner) What is $L_g$?
Only known for $g=2$ (Ham-Song, Cho-Ham). Many other related special cases (Ko-Los-Song, Ham-Song, Lanneau-Thiffeault, Boissy-Lanneau, Liechti-Strenner)

Question (McMullen) Does $\{gL_g\}_{g=2}^\infty$ converge?

To understand/approach these, need to understand where "small stretch factor" pA's come from.
$\uparrow$ $\log(\lambda(f)) \leq \frac{\text{const}}{g}$

Unifying construction due to McMullen.

Let $f: S \to S$ be any pA, $M_f$ = mapping torus $= S\times[0,1]/(x,1)\sim(f(x),0)$.


$\psi_s$ suspension flow on $M$,

Locally $\psi_s(x,t)=(x,t+s)$.

1st return map of $\psi_s$ to $S=S\times\{0\}=S\times\{1\}$ is $f$.

Theorem (Thurston-Fried). $f: S\to S$ pA, $M=M_f$, then $\exists$:

① open cone $C\subset H_2(M;\mathbb{R})$ containing $[S]$ and

$\forall\,\alpha\in C_{\mathbb{Z}} := H_2(M;\mathbb{Z})\cap C$ $\exists$ representative surface $S_\alpha$

(so $[S_\alpha]=\alpha$) w/ $S_\alpha \pitchfork \psi_s$ and 1st return map a pA, $f_\alpha: S_\alpha\to S_\alpha$.

② $\overline{\chi}: C\to\mathbb{R}$ linear function st. $\overline{\chi}(\alpha)=\chi(S_\alpha)$ $\forall\,\alpha\in C_{\mathbb{Z}}$, and

③ $h: C\to\mathbb{R}$ continuous convex, homogeneous of degree $-1$ function st.

$h(\alpha)=\log(\lambda(f_\alpha))$

Observe: let $\{\alpha_n\}\subset C_{\mathbb{Z}}$ be a sequence w/ $t_n\alpha_n\to[S]$, $\alpha_n$ primitive and distinct (so $S_{\alpha_n}$ connected, $t_n\to 0$)

Ⅰ $\overline{\chi}(S) \longleftarrow \overline{\chi}(t_n\alpha_n)=t_n\chi(S_{\alpha_n})$ so $\chi(S_{\alpha_n})\to\infty$
$\Rightarrow$ genus$(S_{\alpha_n})=g_n\to\infty$

$\overline{\chi}(S) = \chi(S)$

Ⅱ $h(\alpha_n)|\overline{\chi}(\alpha_n)| = \log(\lambda(f_{\alpha_n}))|\chi(S_{\alpha_n})| = \underbrace{\log(\lambda(f_{\alpha_n}))(2g_n-2)}$

$\frac{1}{t_n}h(\alpha_n)\,t_n|\overline{\chi}(\alpha_n)| = h(t_n\alpha_n)|\overline{\chi}(t_n\alpha_n)| \longrightarrow \log(\lambda(f))|\chi(S)| = \text{constant}.$

Theorem (Farb-L-Margalit) Any sequence $\{f_n: S_{g_n}\to S_{g_n}\}_{n=1}^{\infty}$ w/ $g_n\log(\lambda(f_n))\le C$ comes from above construction, up to subsequence and puncturing at cone pts.