Mirzakhani's count of simple closed geodesics

Anton Zorich

Summer school "Teichmüller dynamics,
mapping class groups and applications"

Grenoble, June 21, 2018

Space of multicurves

Train tracks carrying simple closed curves

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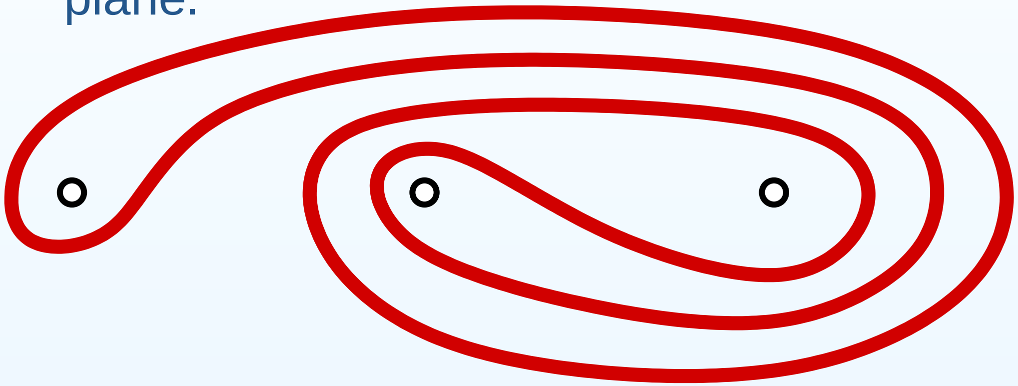

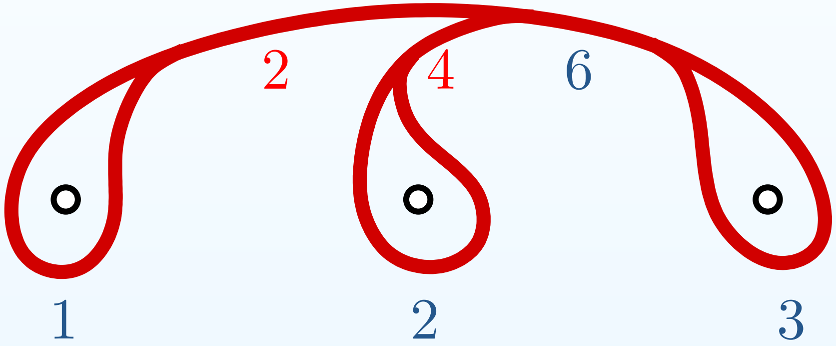

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Each edge of the graph τ is the smooth image of an interval; at each vertex of τ (called "*switch*") there is a well-defined tangent line; the integer weights (recording the number of strands) satisfy the switch condition at each switch: the sums of the weights on each side of the switch are equal to each other.

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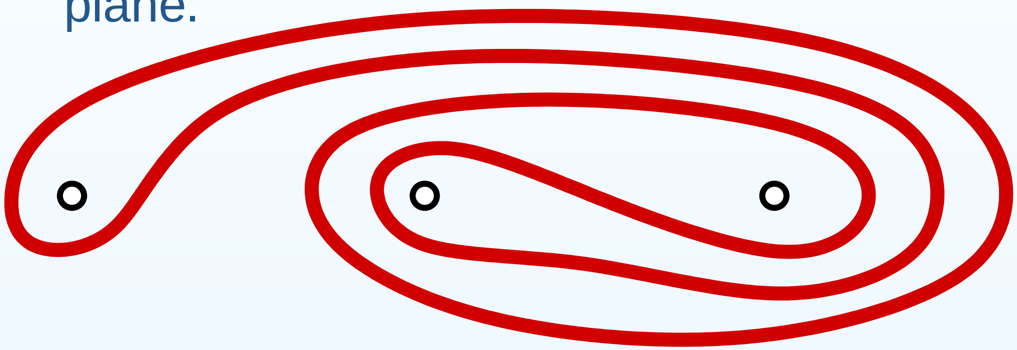

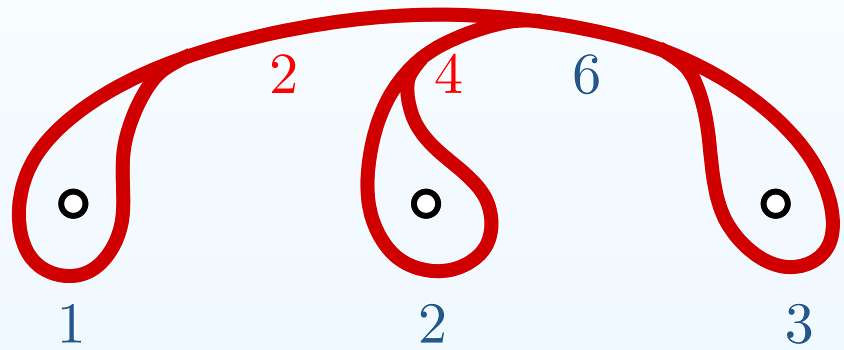

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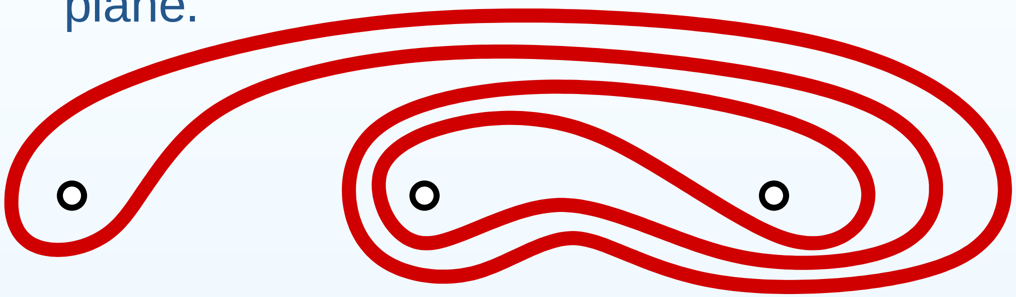

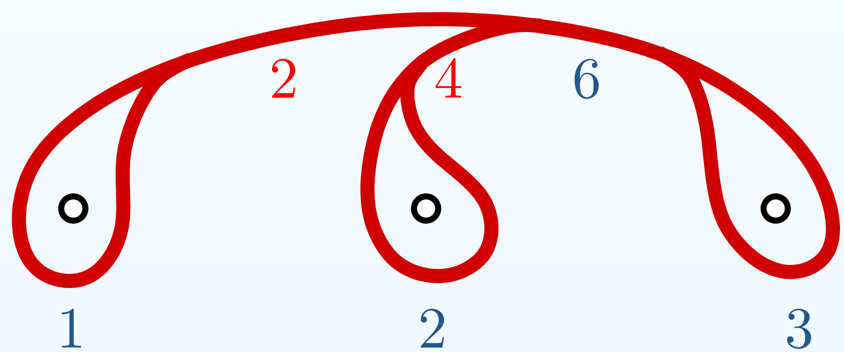

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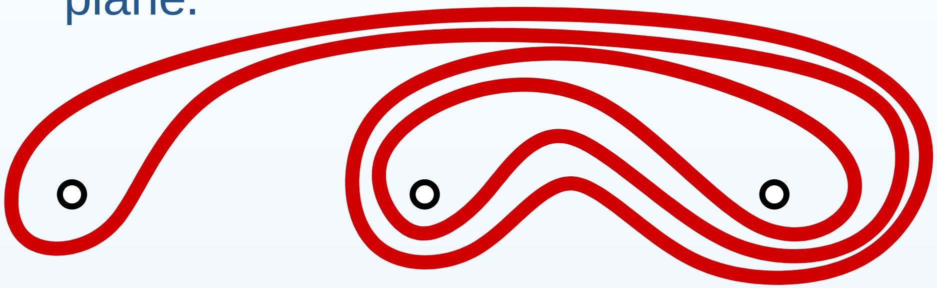

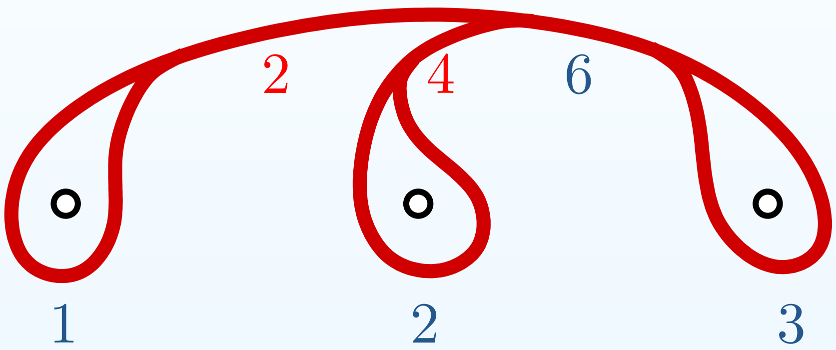

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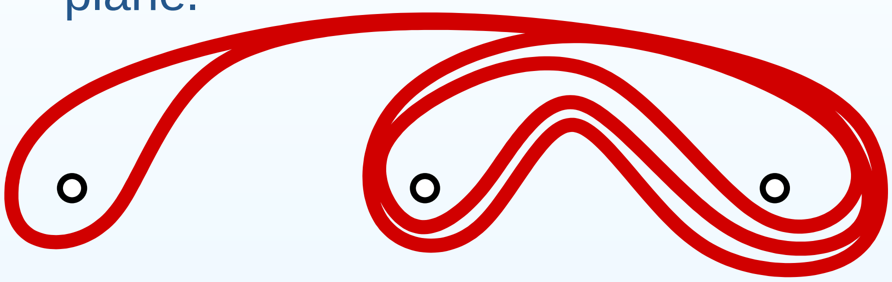

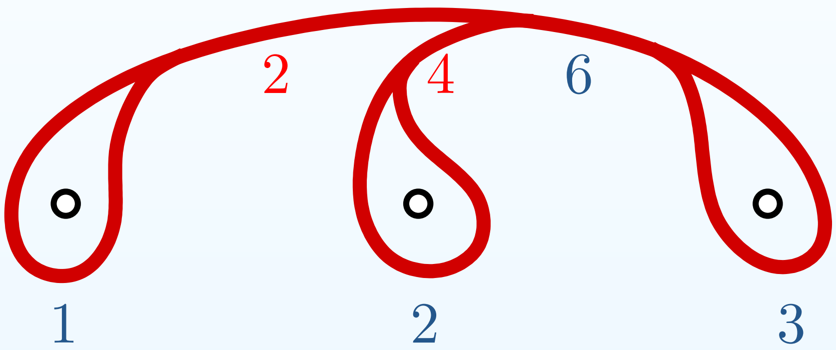

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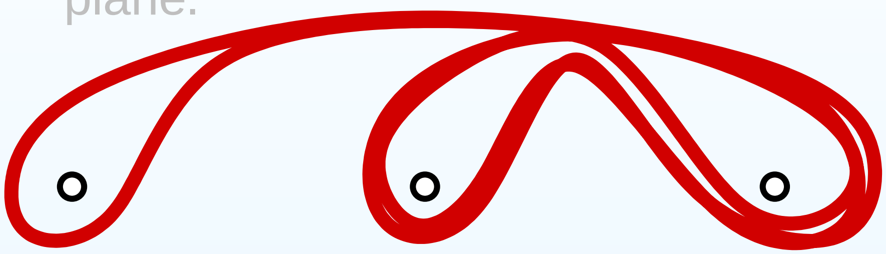

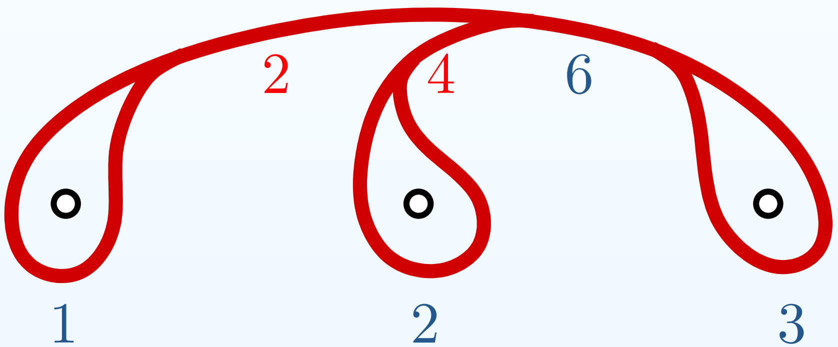

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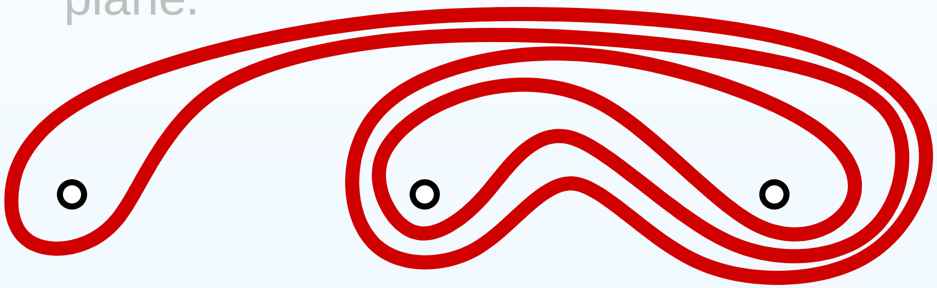

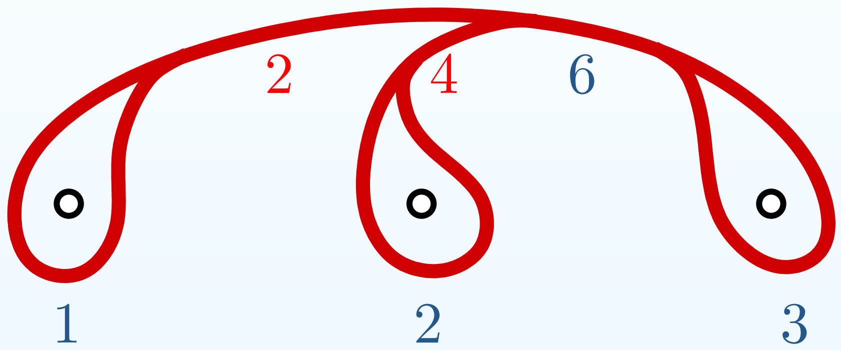

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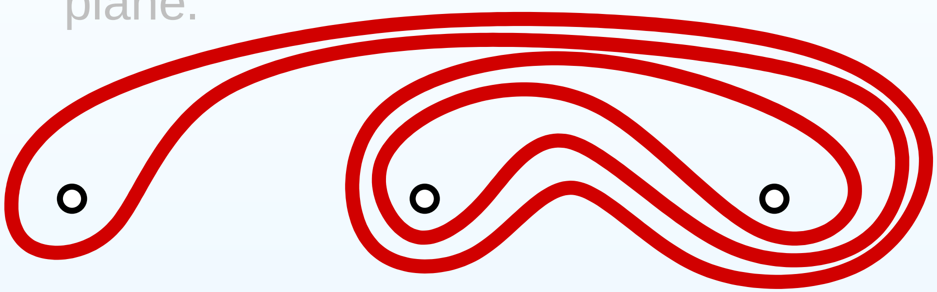

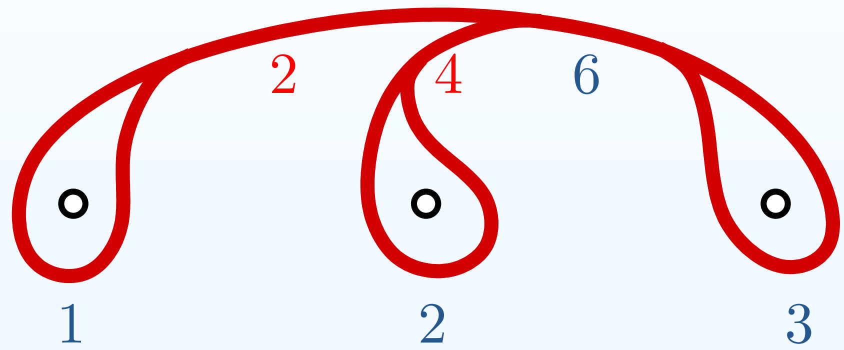

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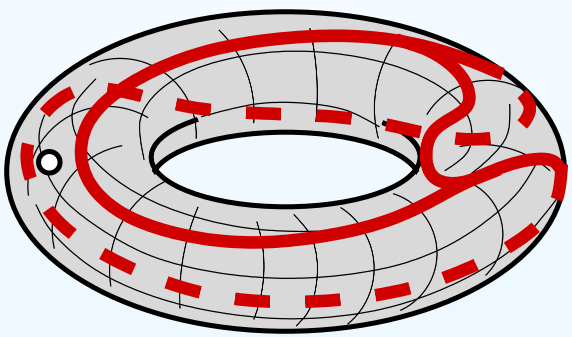
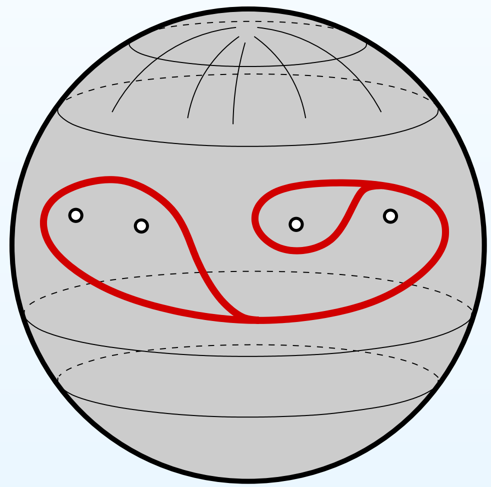
Exercise on train-tracks

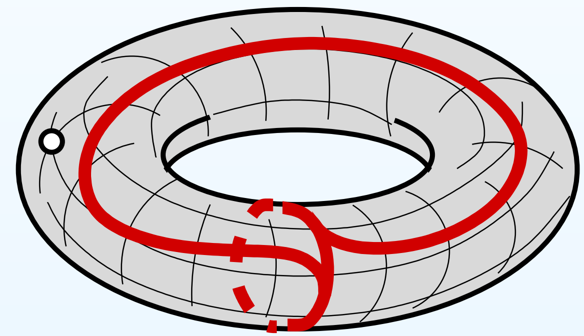
Which of the given train-tracks τ_1, τ_2, τ_3 might carry a simple closed hyperbolic geodesic? Indicate some legitimate weights if you claim that the train track carries a simple closed hyperbolic geodesic.



τ_1



τ_2



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Can any of the given train-tracks τ_1, τ_2, τ_3 carry *different* simple closed hyperbolic geodesic? Indicate the corresponding different legitimate collections of weights if you claim that.

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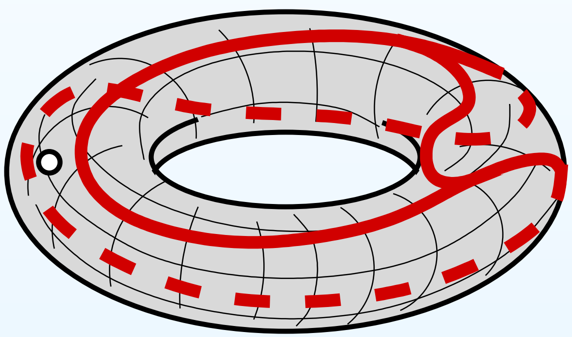

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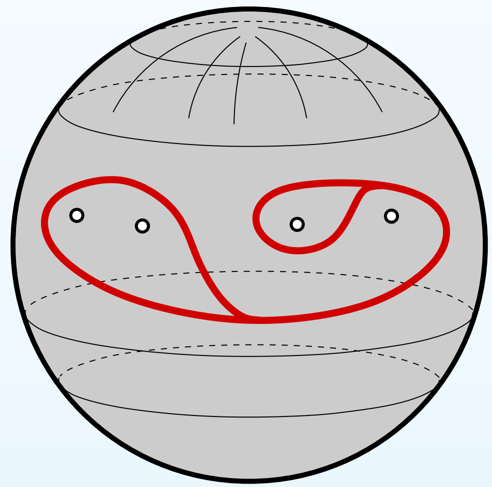
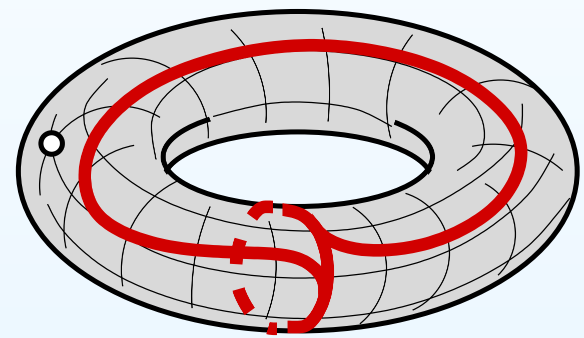


τ_2



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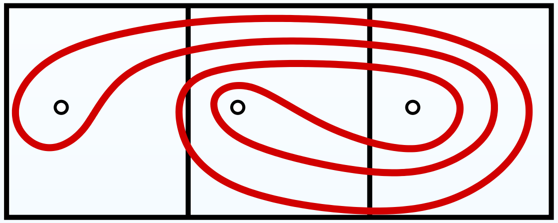
Four basic train tracks on $S_{0,4}$

Up to isotopy, any simple closed curve in $S_{0,4}$ can be drawn inside the three squares:



By further isotopy, we eliminate bigons with the vertical edges of the three squares. Each connected component of the intersection of γ with the corresponding square is now one of the six types of arcs shown at the right picture. Since γ is essential, it cannot use both types of horizontal segments. Since the other two types of arcs in the middle square intersect, γ can use at most one of those.

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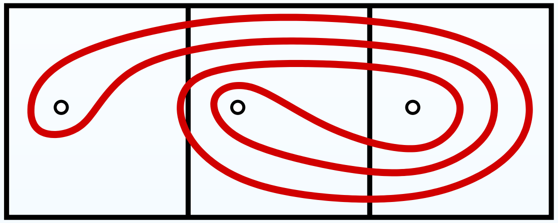

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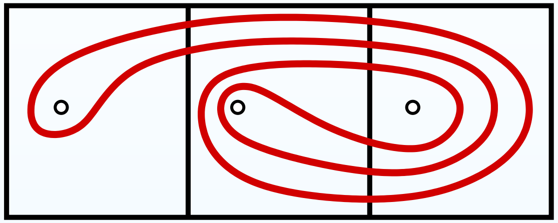

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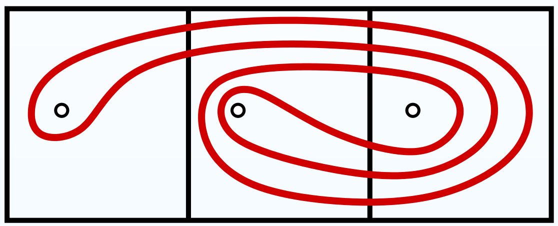

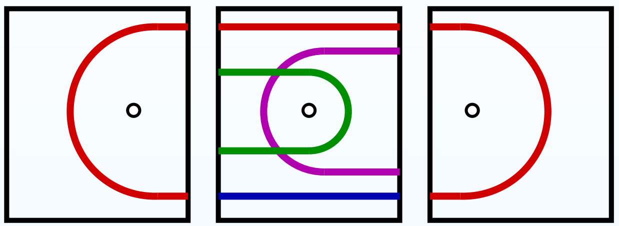

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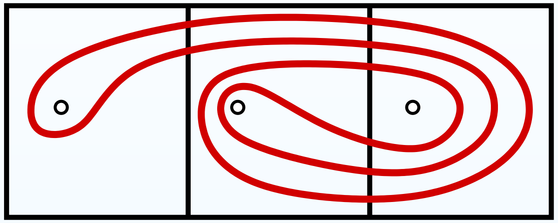

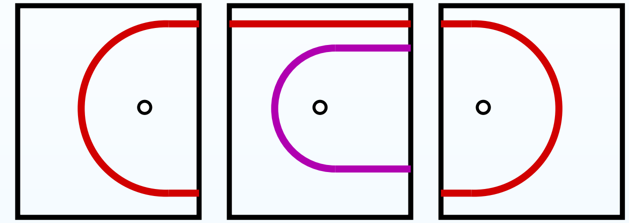

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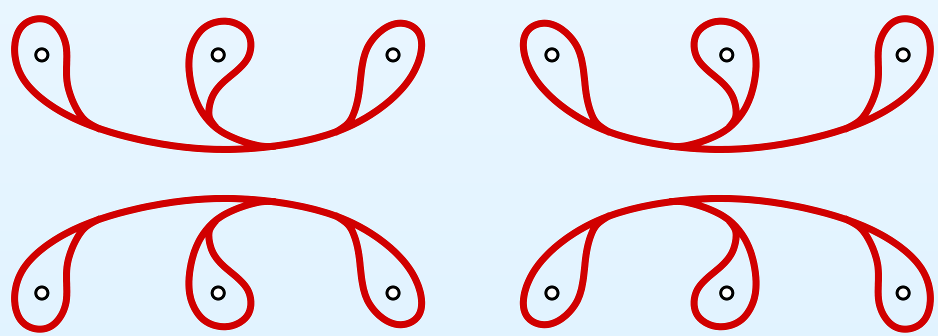
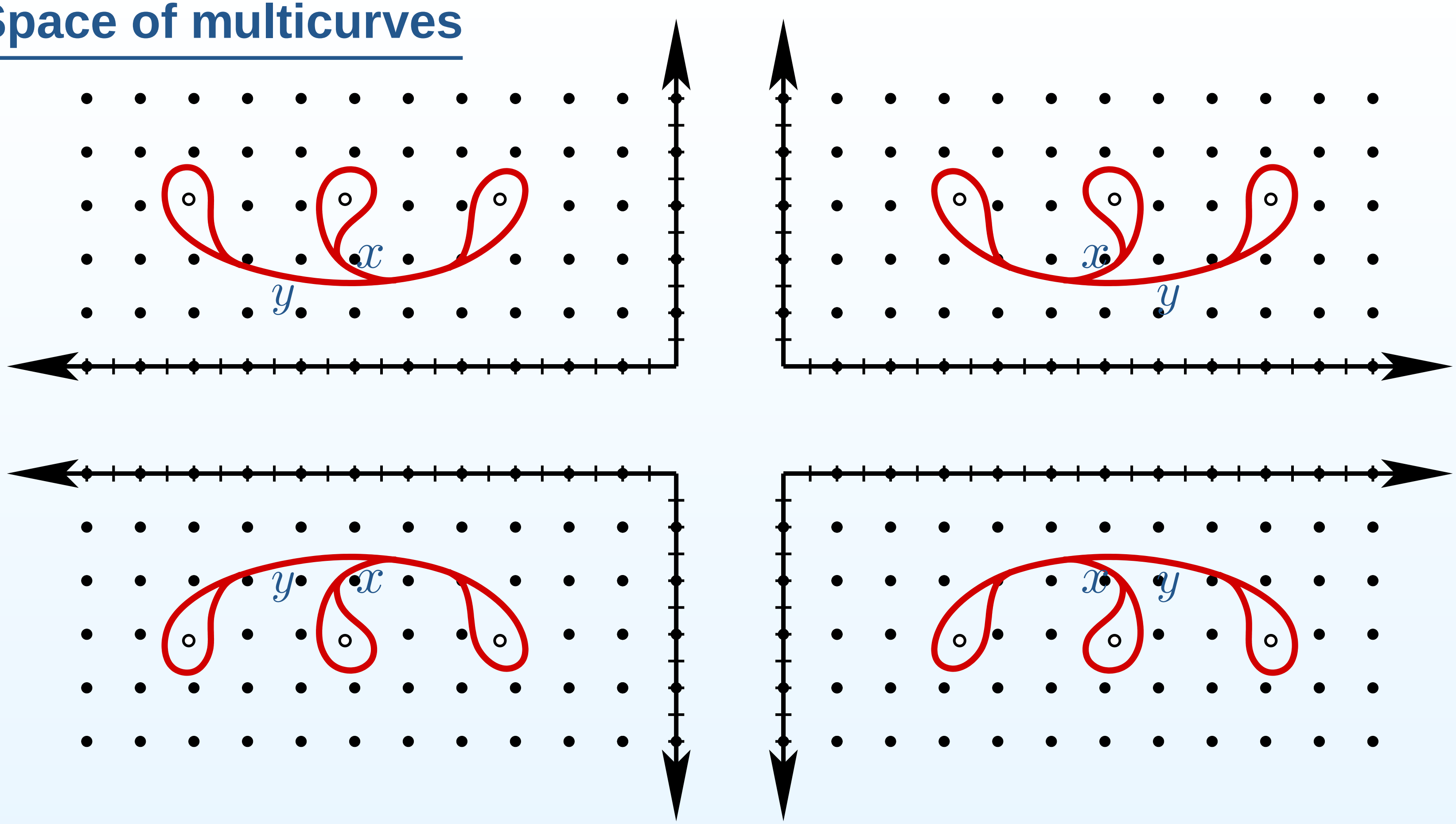
Conclusion: there are four types of simple closed curves in $S_{0,4}$, depending on which of each of the two pairs of arcs they use in the middle square. This is the same as saying that any simple closed curve is carried by one of the following four train tracks:



Space of multicurves



The four train tracks τ_1, τ_2, τ_3, τ_4 give four coordinate charts on the set of isotopy classes of simple closed curves in $S_{0,4}$. Each coordinate patch corresponding to a train track τ_i is given by the weights (x, y) of two chosen edges of τ_i. If we allow the coordinates x and y to be arbitrary nonnegative real numbers, then we obtain for each τ_i a closed quadrant in $\mathbb{R}^2$. Arbitrary points in this quadrant are measured train tracks.

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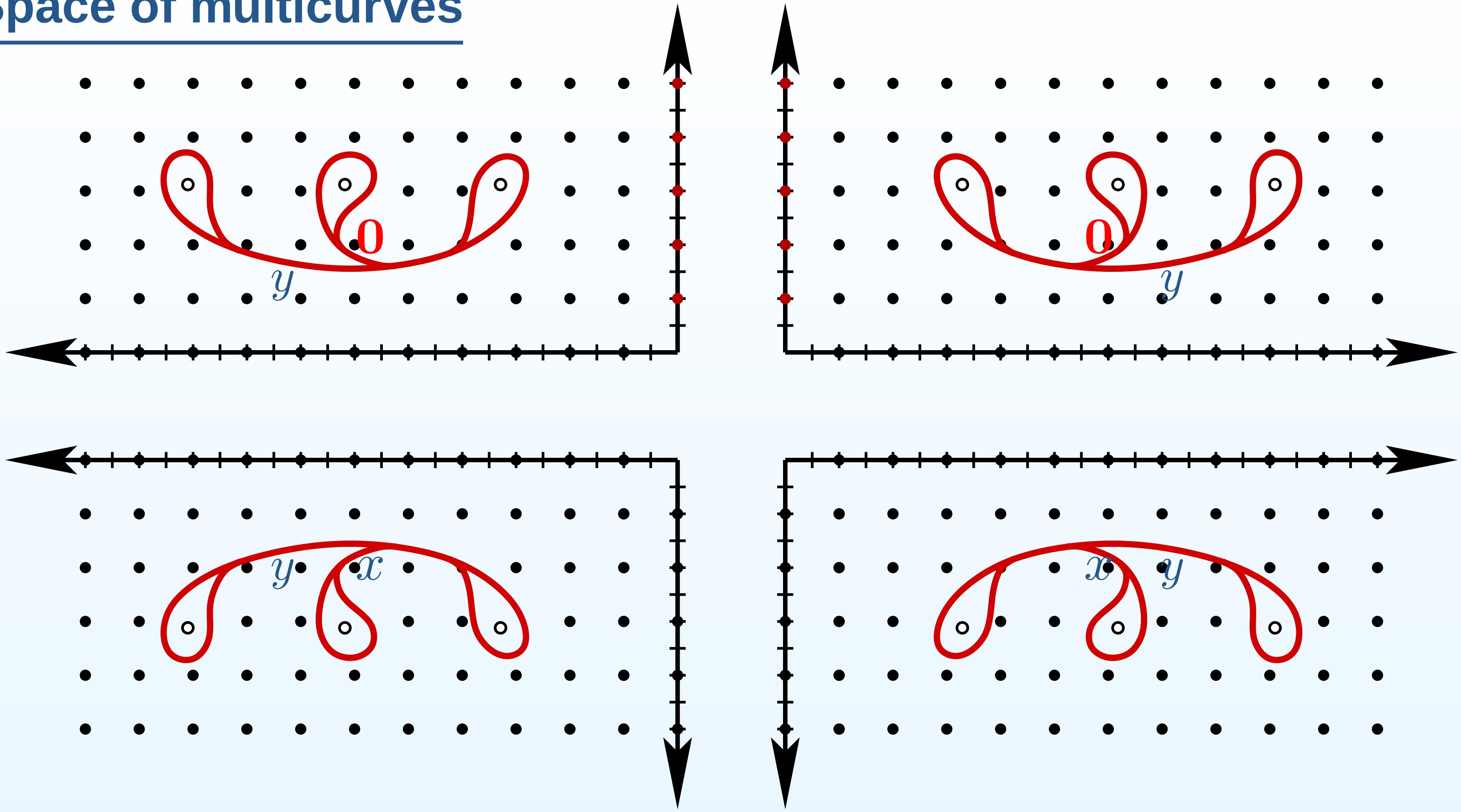

Weight zero on an edge of a train track tells that such edge can be deleted. This implies that pairs of quadrants should be identified along their edges.

The resulting space is homeomorphic to $\mathbb{R}^2$. The integral points in this $\mathbb{R}^2$ correspond to isotopy classes of multicurves in $S_{0,4}$.

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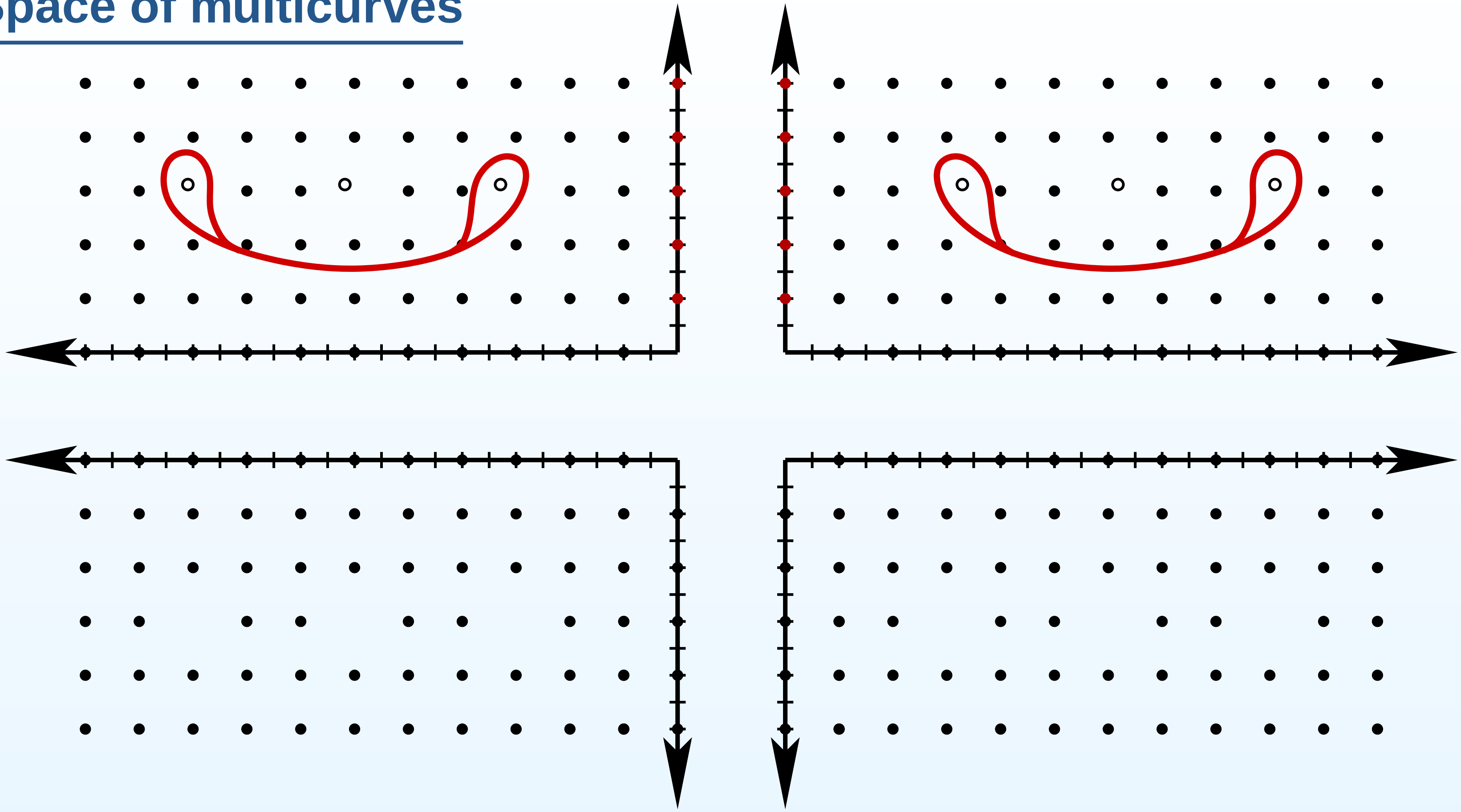

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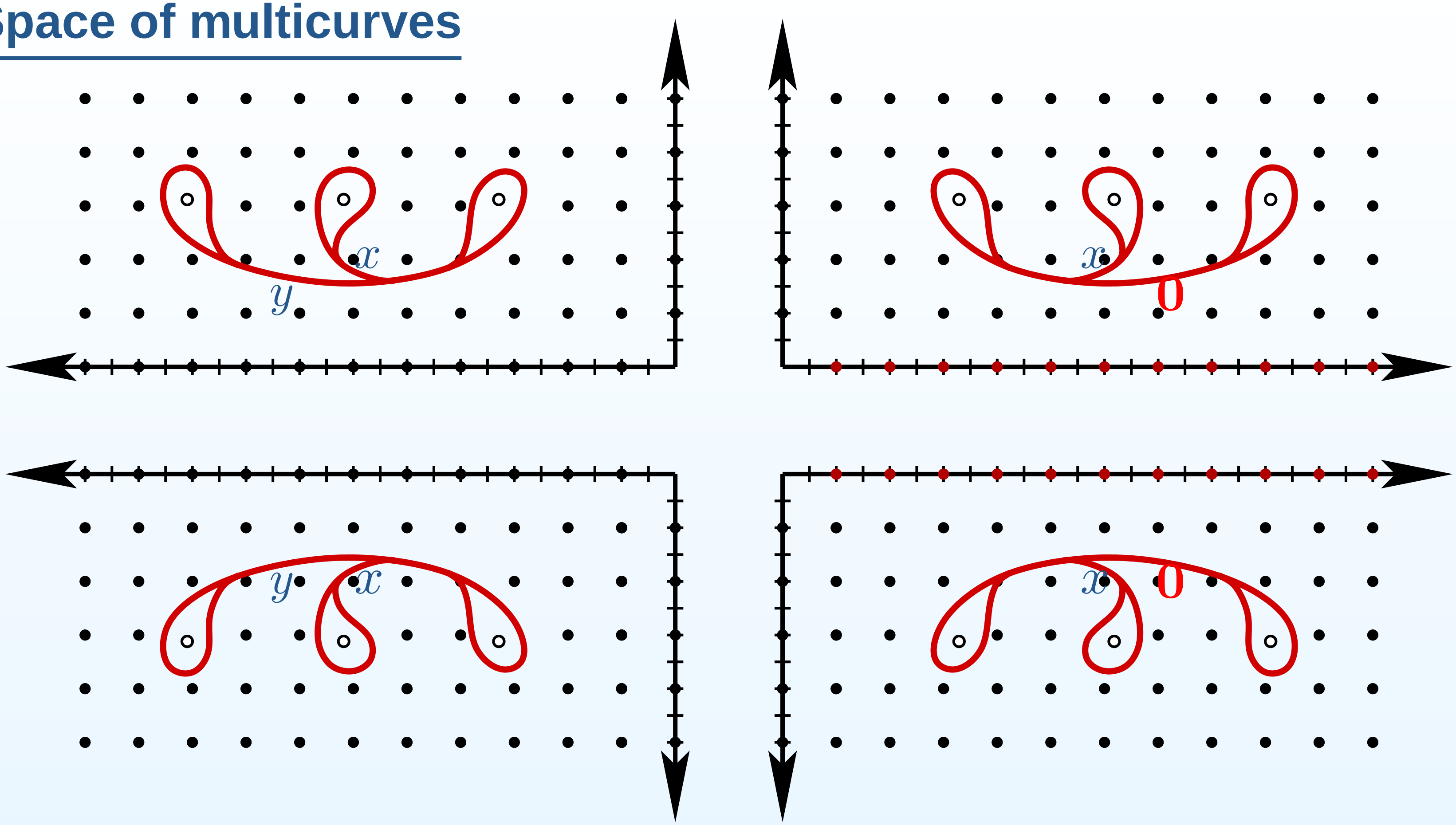

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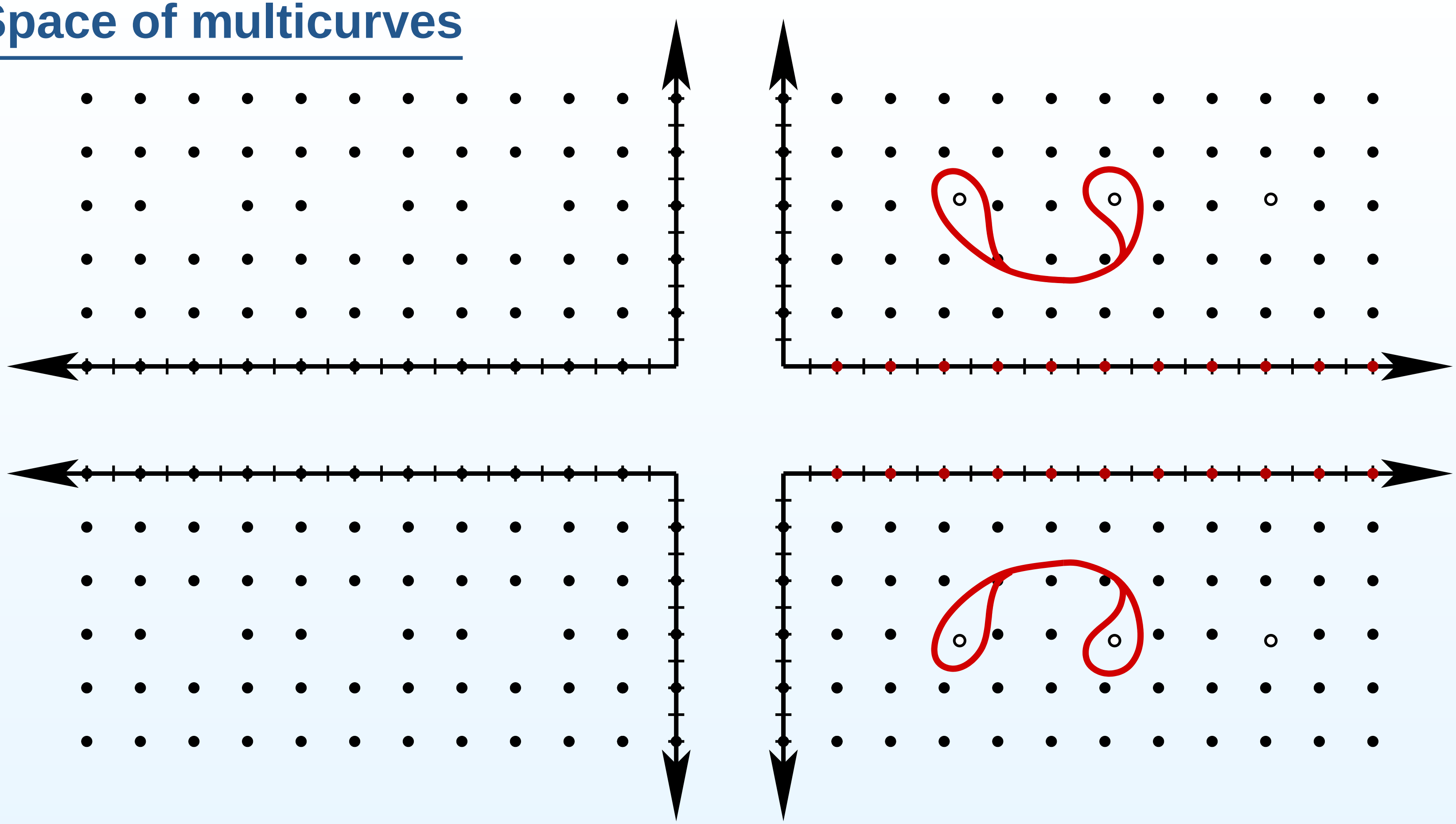

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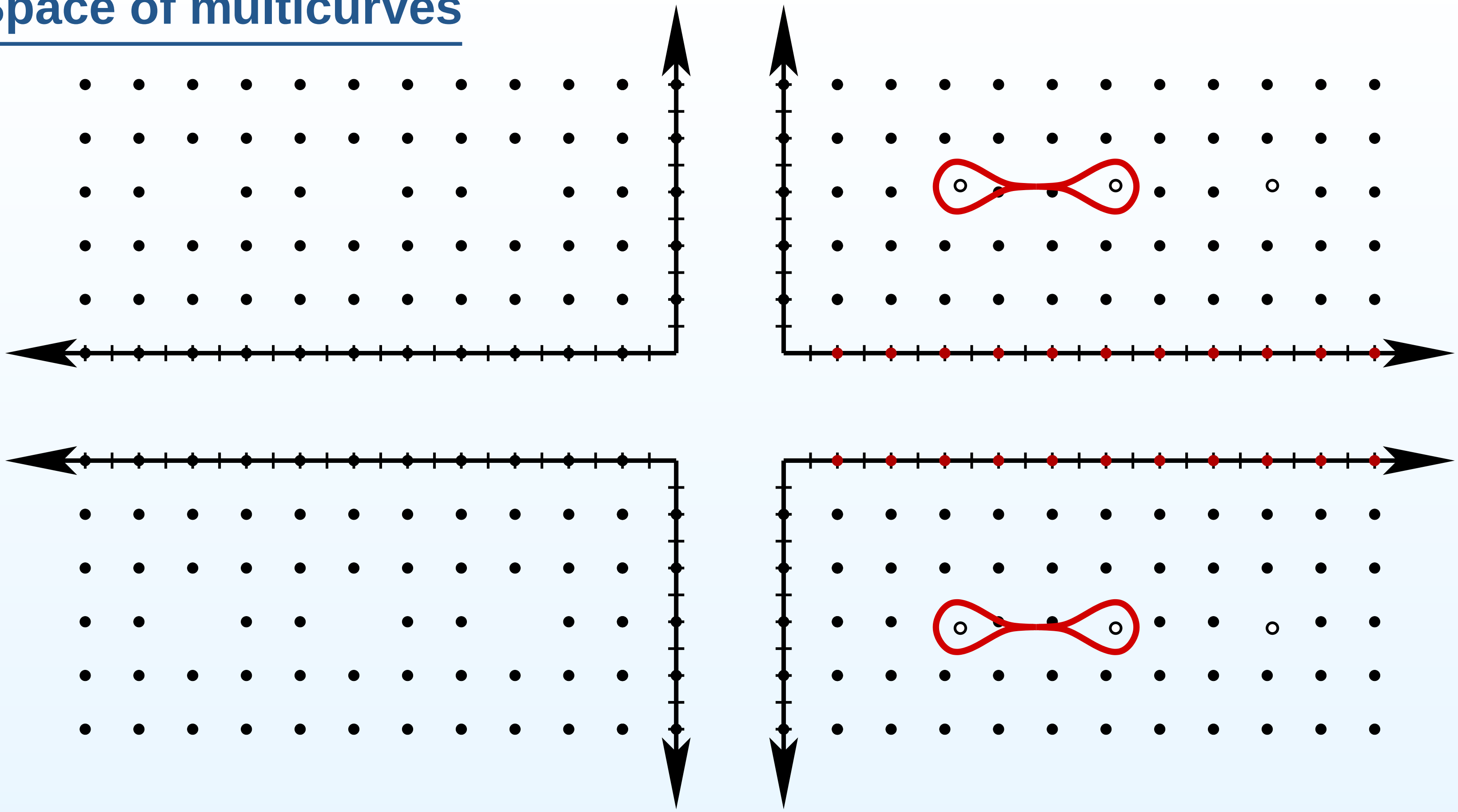

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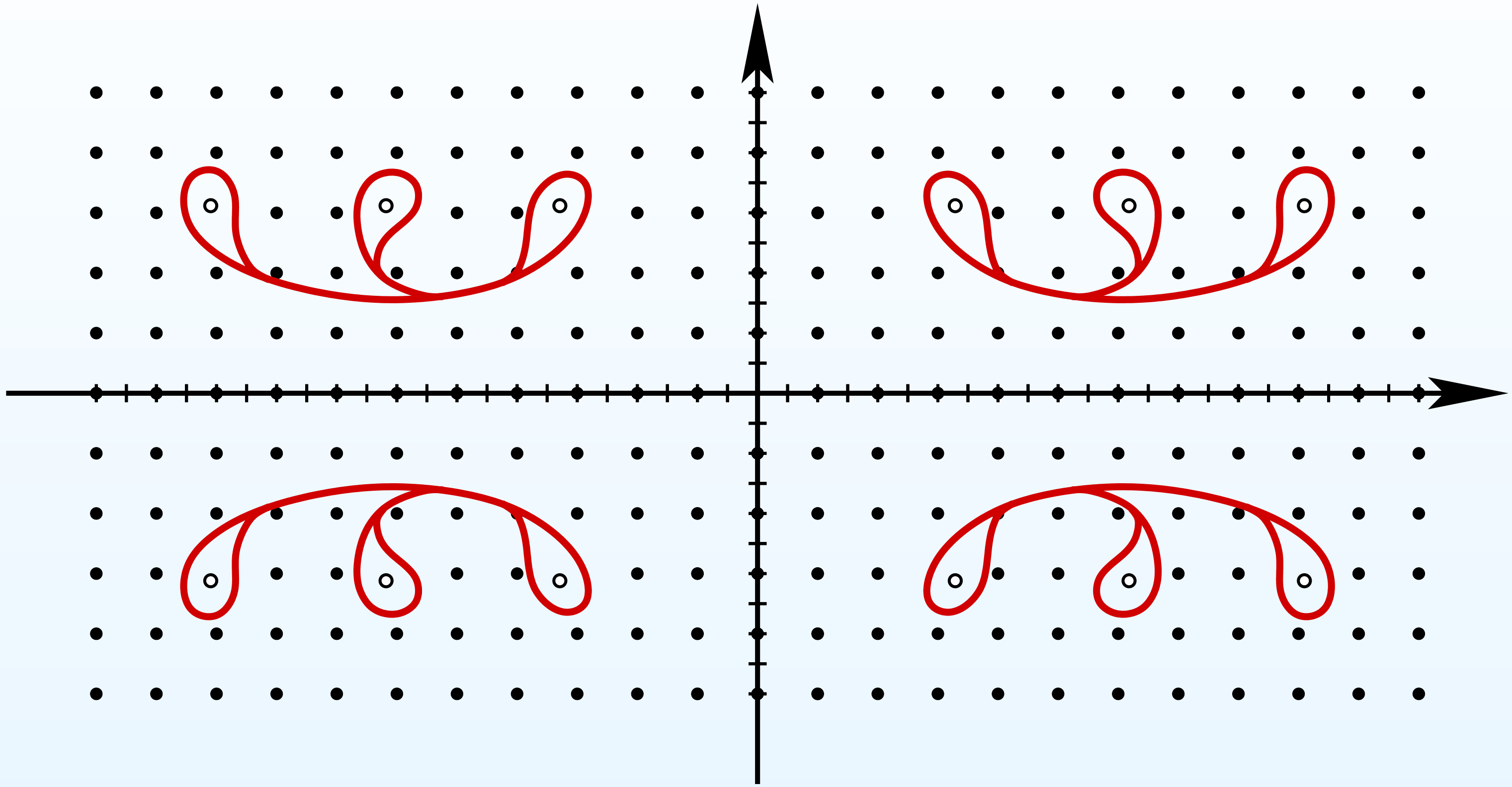

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Thurston's and Mirzakhani's measures on $\mathcal{ML}_{g,n}$

Space $\mathcal{ML}_{g,n}$ and the length function

Similar considerations applied to a smooth surface $S_{g,n}$ lead to analogous space $\mathcal{ML}_{g,n}$ endowed with a **piecewise linear structure**.

Up to now we did not use hyperbolic metric on $S_{g,n}$. In the presence of a hyperbolic metric, integral points of $\mathcal{ML}_{g,n}$ can be interpreted as simple closed geodesic multicurves.

Moreover: all other points also get geometric realization as *measured geodesic laminations* — disjoint unions of non self-intersecting infinite geodesics.

The hyperbolic length $\ell_\gamma(X)$ of a simple closed geodesic γ on a hyperbolic surface $X \in \mathcal{T}_{g,n}$ determines a real analytic function on the Teichmüller space.

One can extend the length function by linearity to simple closed multicurves:

$$\ell_{\sum a_i\gamma_i} := \sum a_i \ell_{\gamma_i}(X)\,.$$

By homogeneity and continuity the length function can be further extended to $\mathcal{ML}_{g,n}$. By construction $\ell_{t\cdot\lambda}(X) = t\cdot\ell_\lambda(X)$.

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Thurston measure on $\mathcal{ML}_{g,n}$

Let us temporarily return to the situation when there is no hyperbolic metric on a smooth surface $S_{g,n}$.

Train track charts define **piecewise linear structure** on $\mathcal{ML}_{g,n}$.

“Integral lattice” $\mathcal{ML}_{g,n}(\mathbb{Z})$ provides canonical normalization of the linear volume form μ_{Th} in which the fundamental domain of the lattice has unit volume.

Integral points in $\mathcal{ML}_{g,n}$ are in a one-to-one correspondence with the set of integral multi-curves, so the piecewise-linear action of $\mathrm{Mod}_{g,n}$ on $\mathcal{ML}_{g,n}$ preserves the “integral lattice” $\mathcal{ML}_{g,n}(\mathbb{Z})$, and, hence, preserves the measure μ_{Th}.

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Thurston measure on $\mathcal{ML}_{g,n}$

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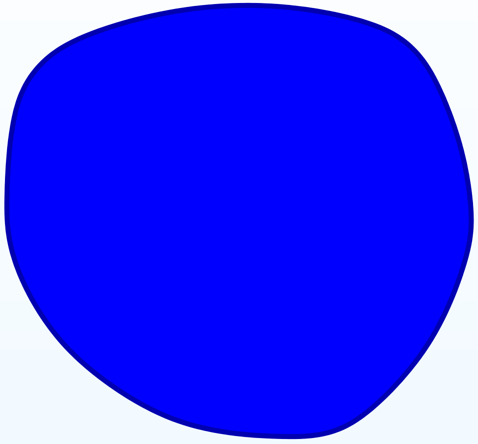
Counting the measure of a set



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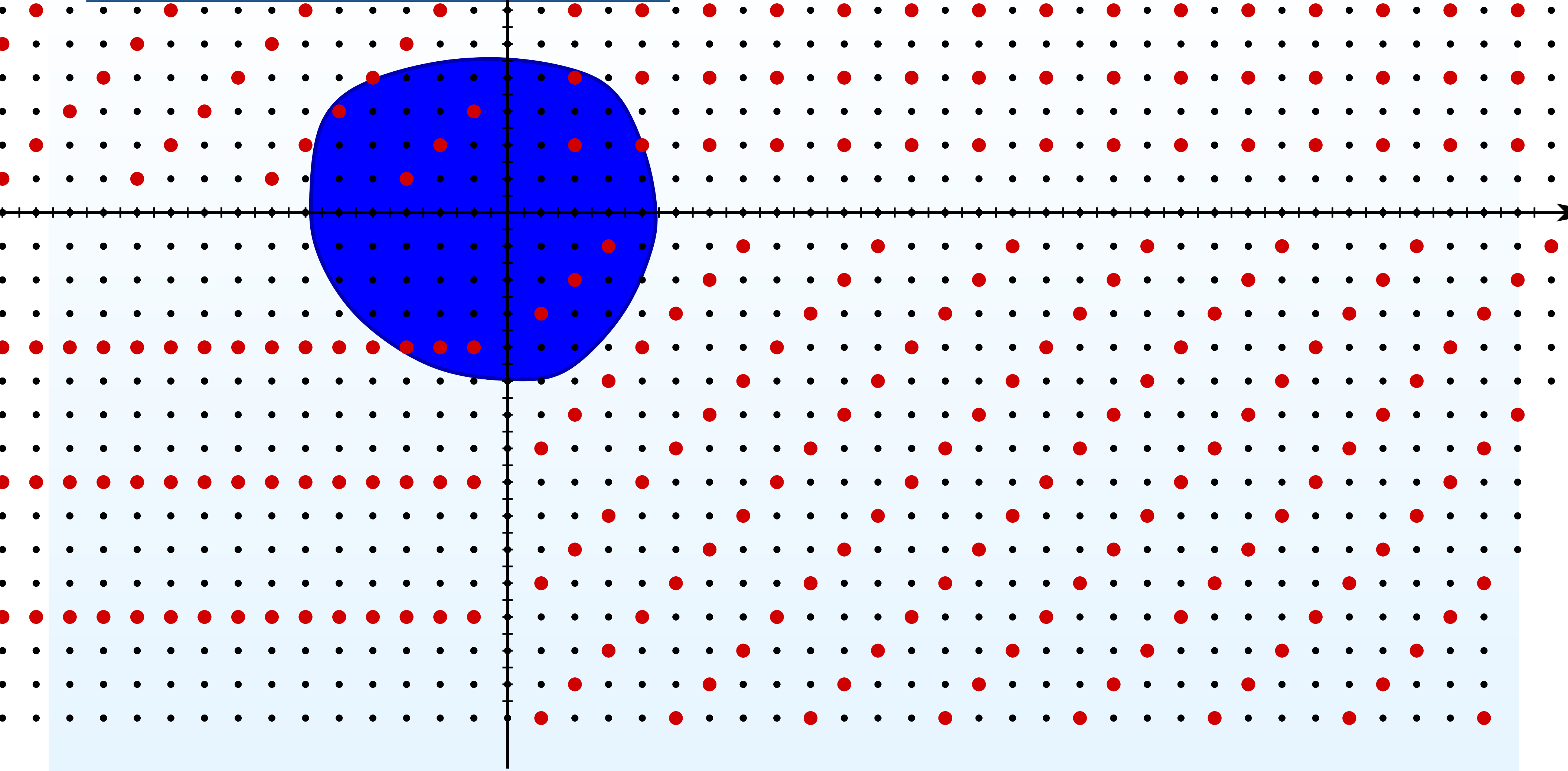

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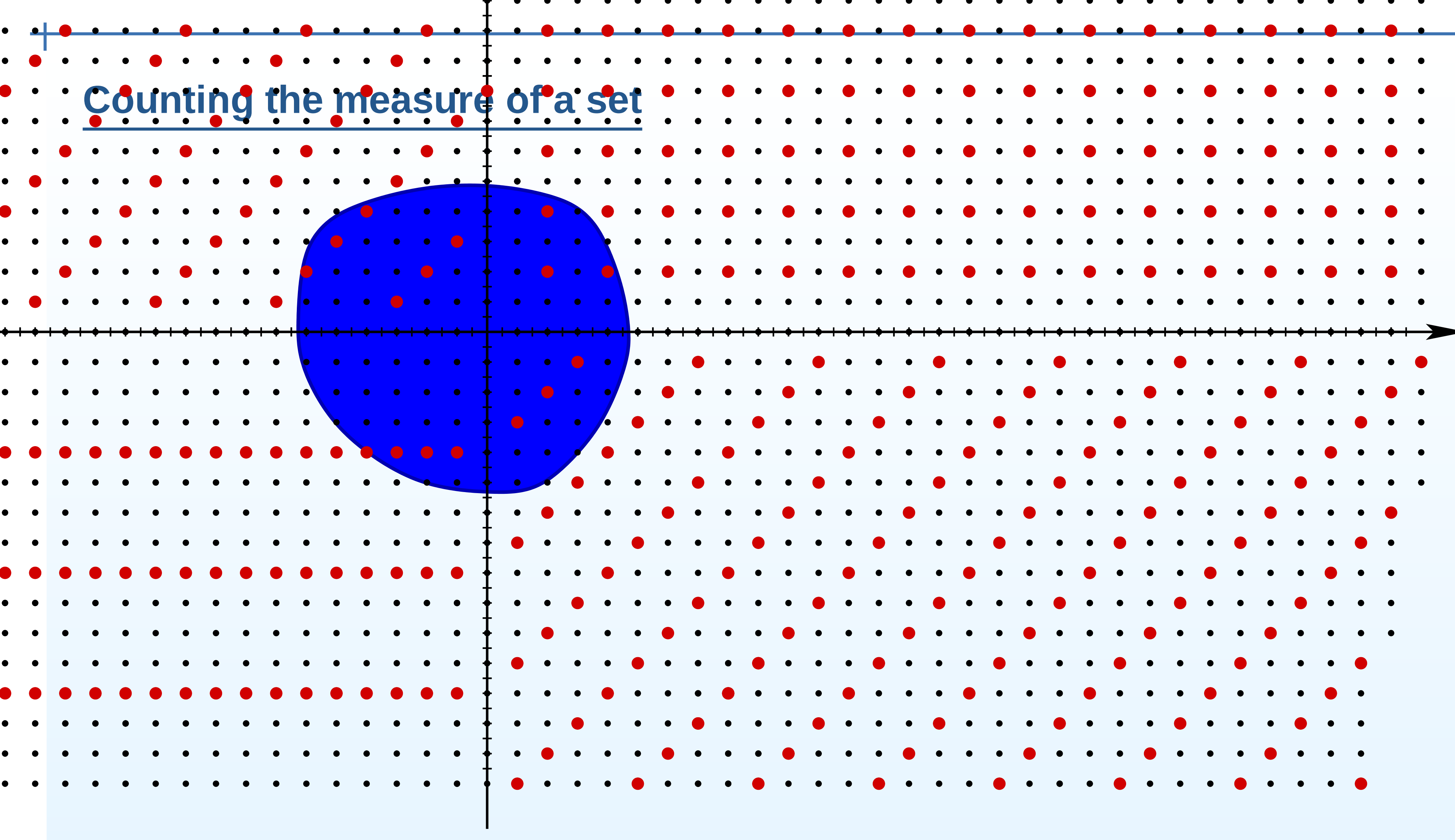

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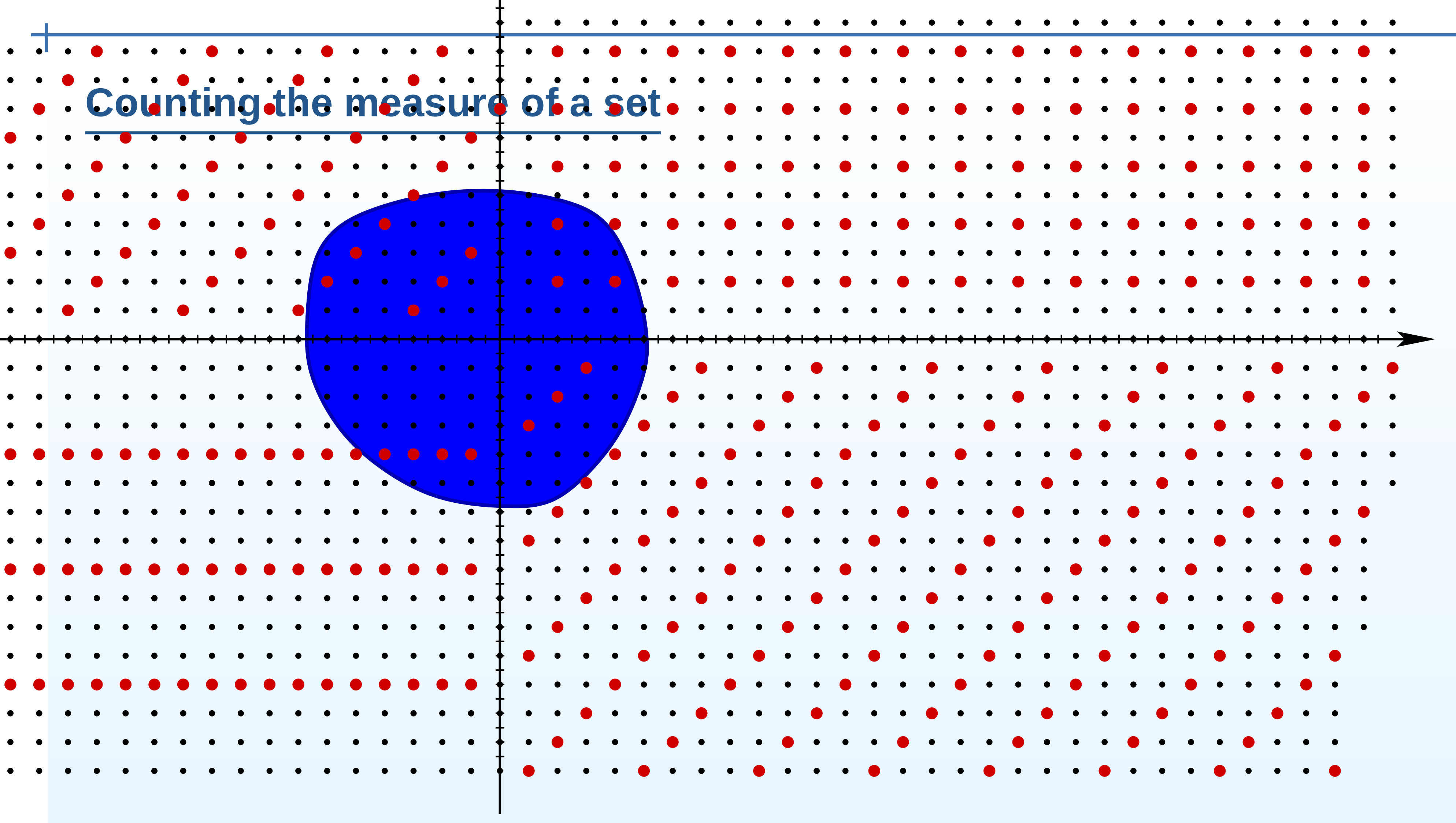

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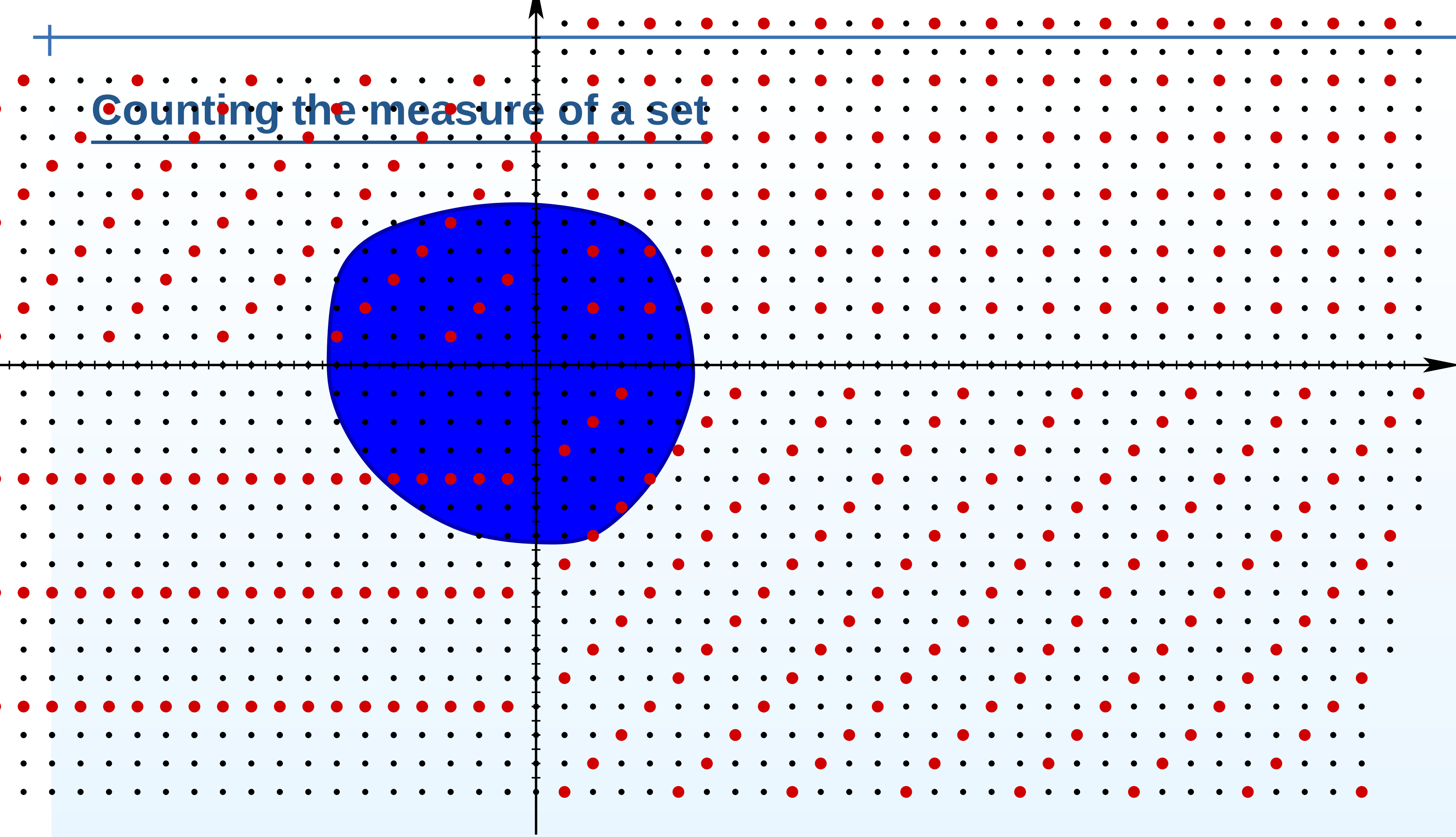

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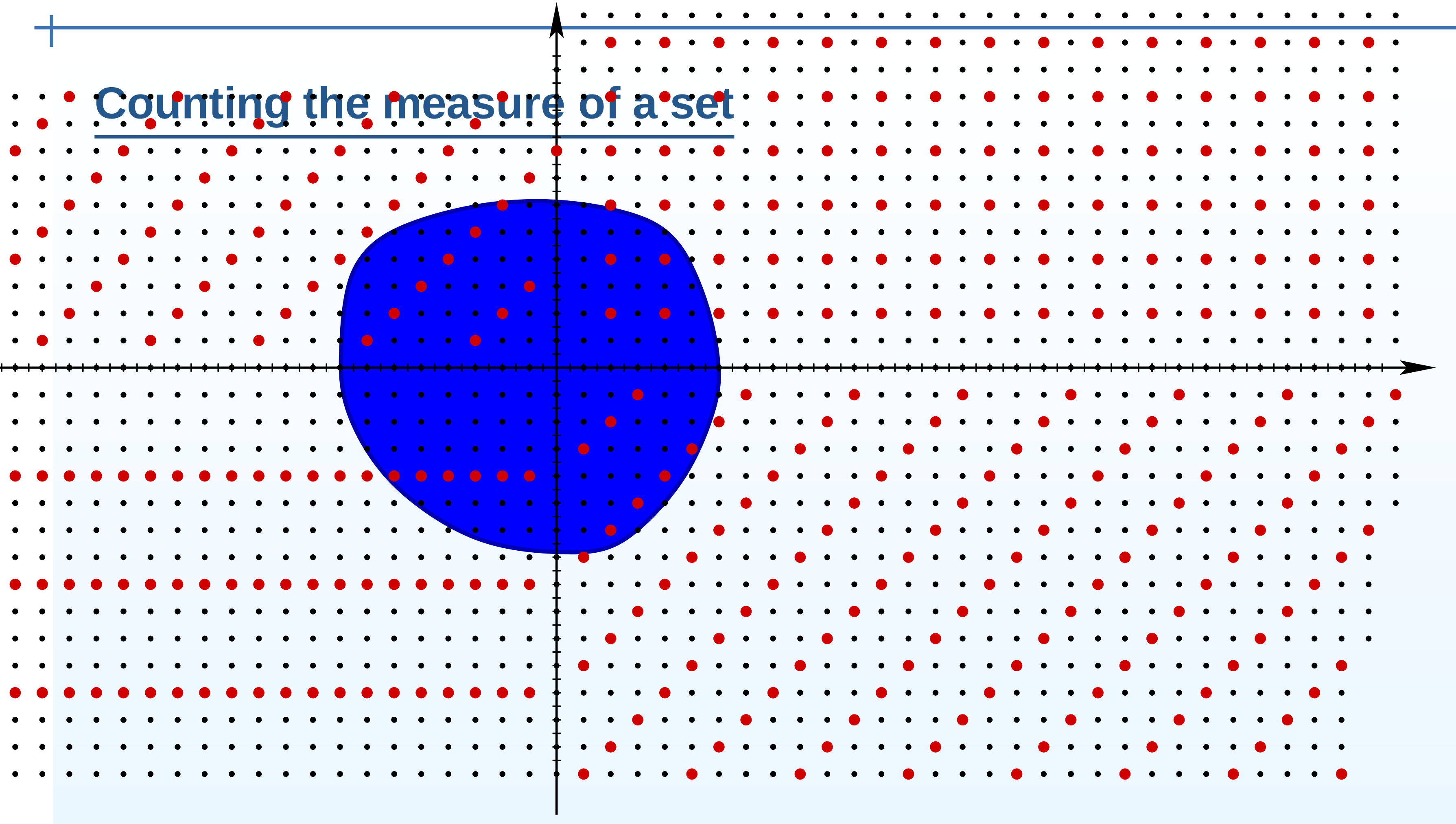

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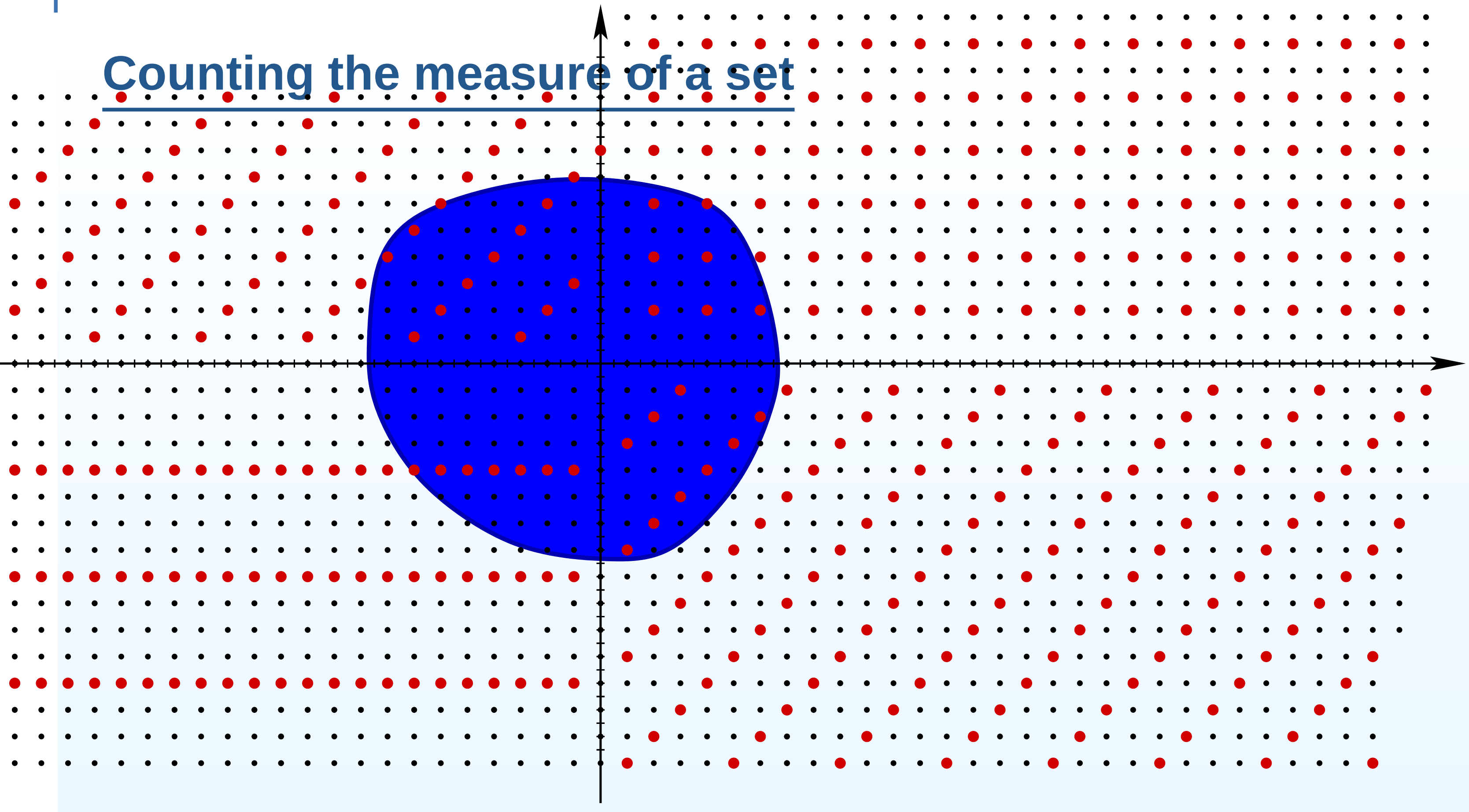

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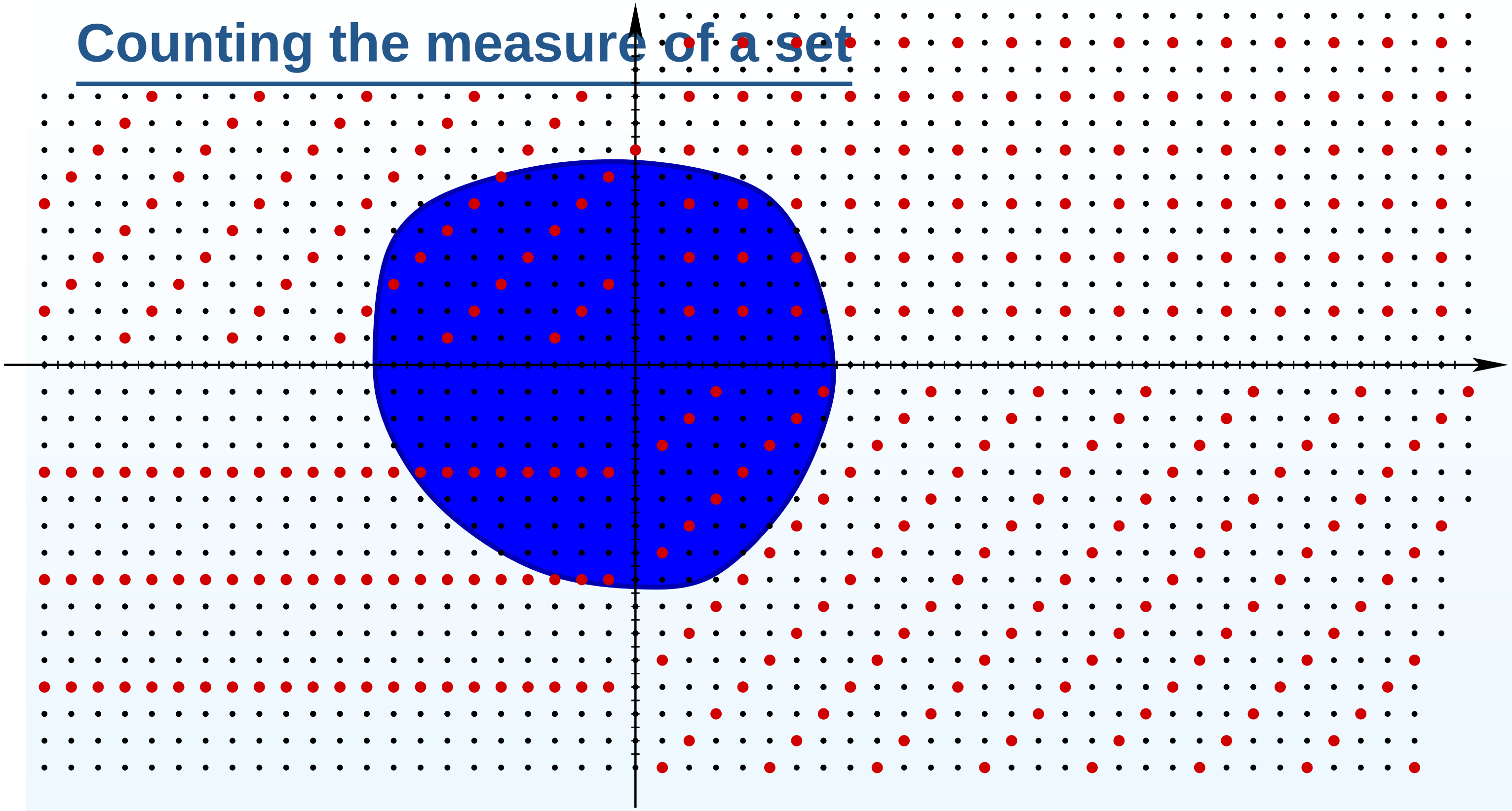

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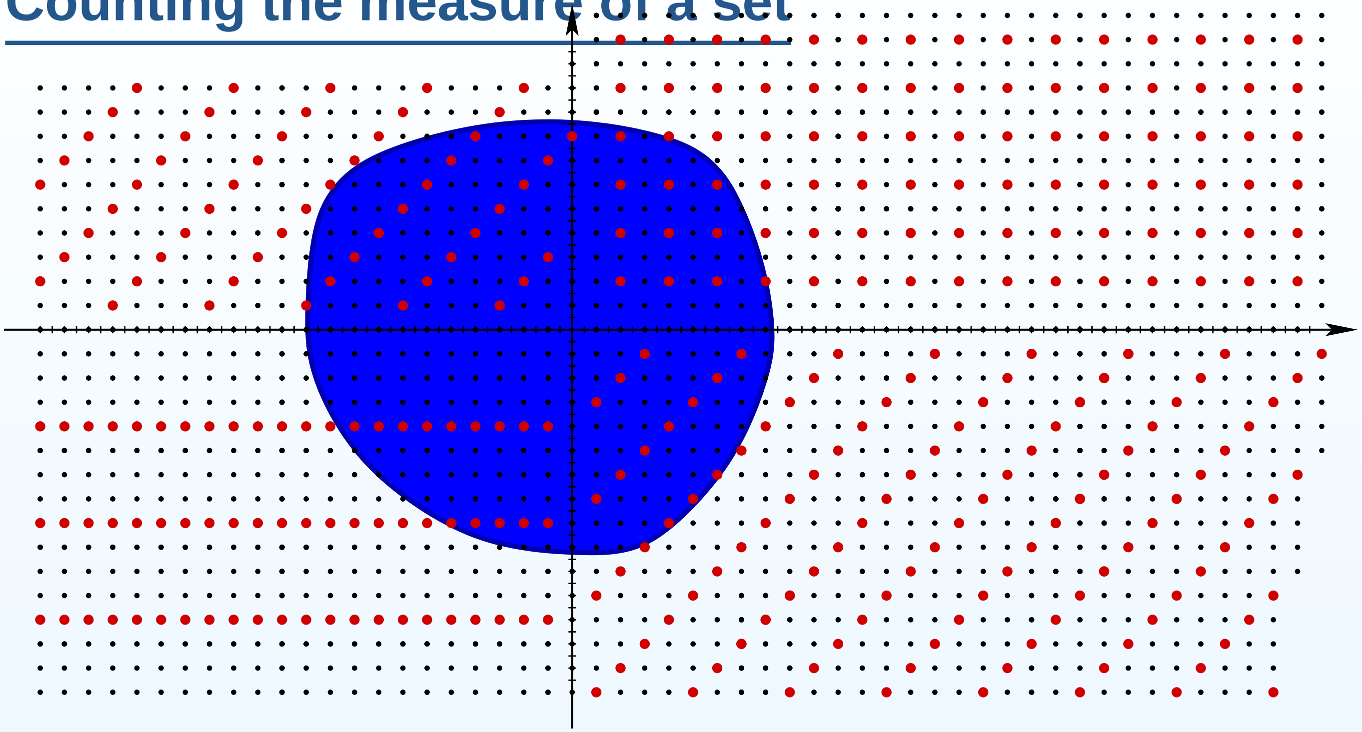

By definition, the Lebesgue measure $\mu(U)$ of a set $U \subset \mathbb{R}^n$ is defined as the limit of the normalized number of points of the ε-grid which get to U:

$$\mu(U) := \lim_{\varepsilon \to 0} \varepsilon^n \cdot \mathrm{card}(U \cap \varepsilon \mathbb{Z}^n)\,.$$

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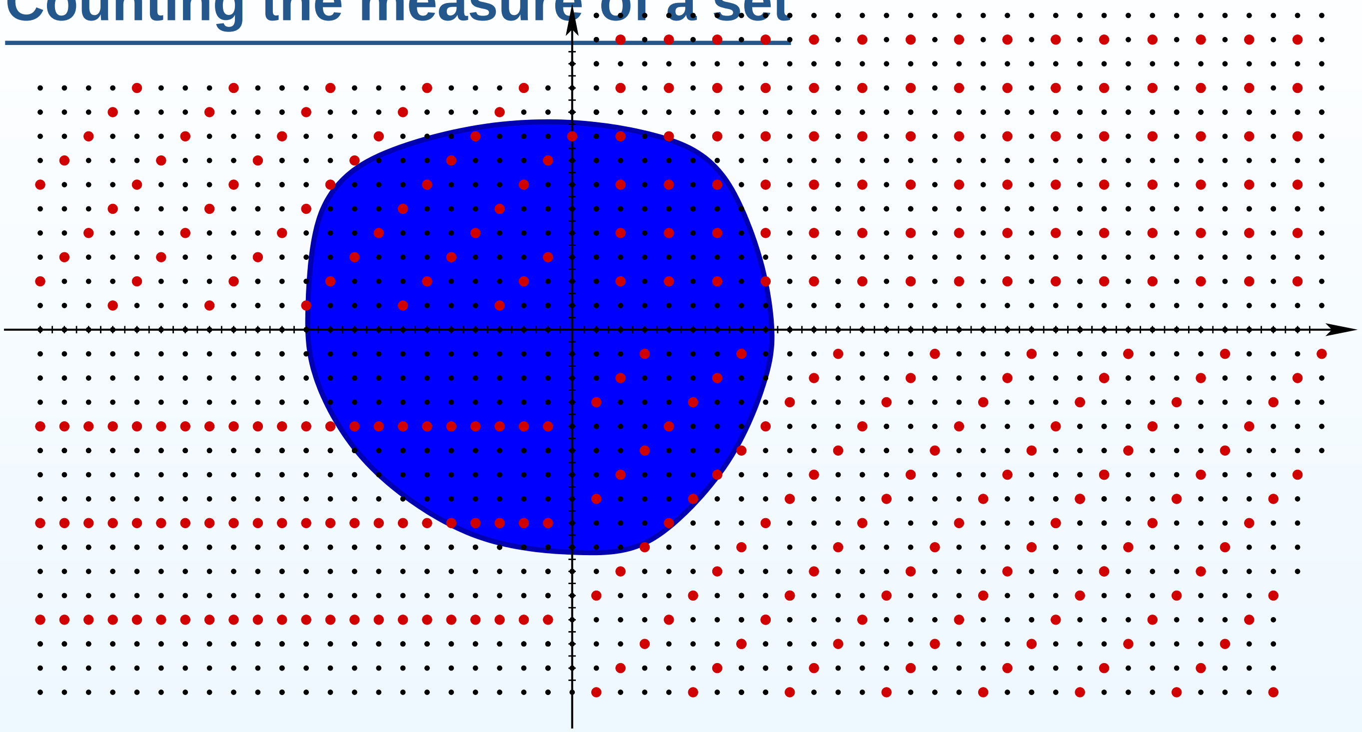

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We can fix U and scale the lattice or can fix the lattice and scale U:

$$\operatorname{card}(U \cap \varepsilon \mathbb{Z}^n) = \operatorname{card}\left(\frac{1}{\varepsilon} U \cap \mathbb{Z}^n\right)$$

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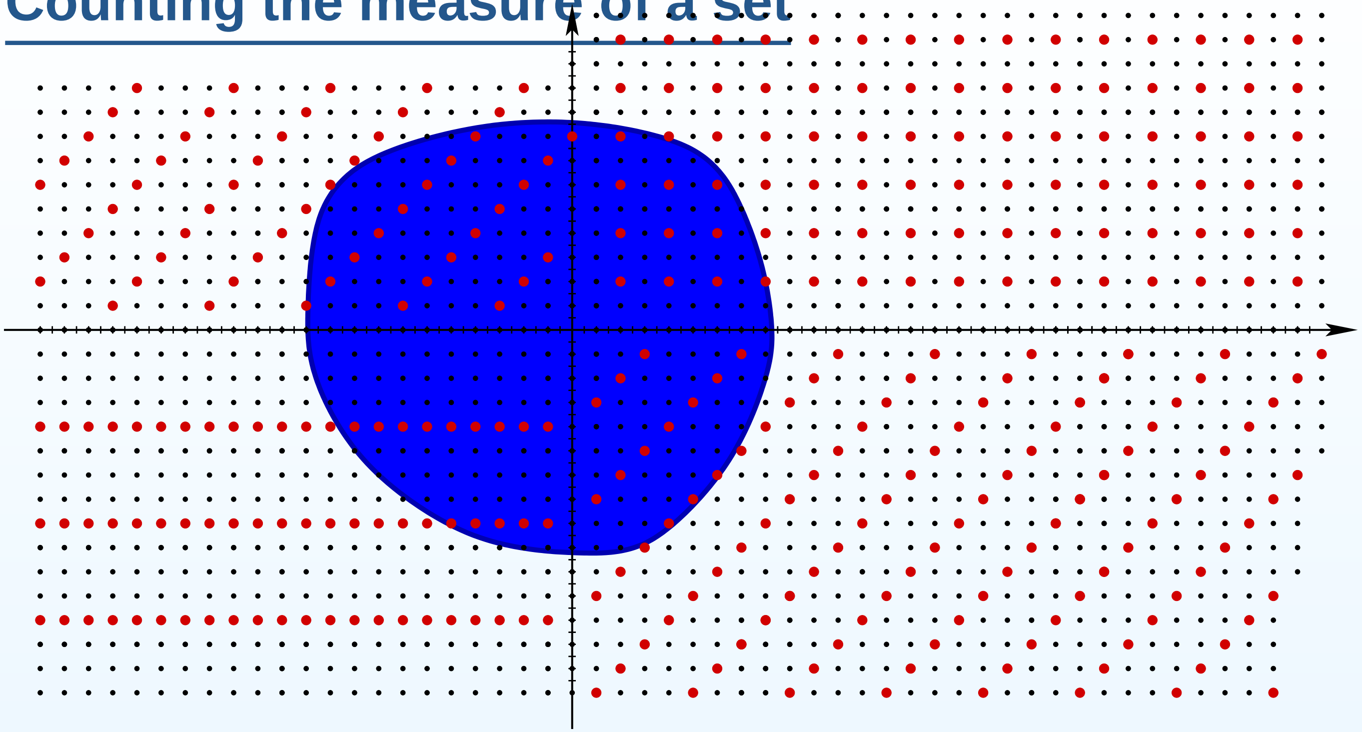
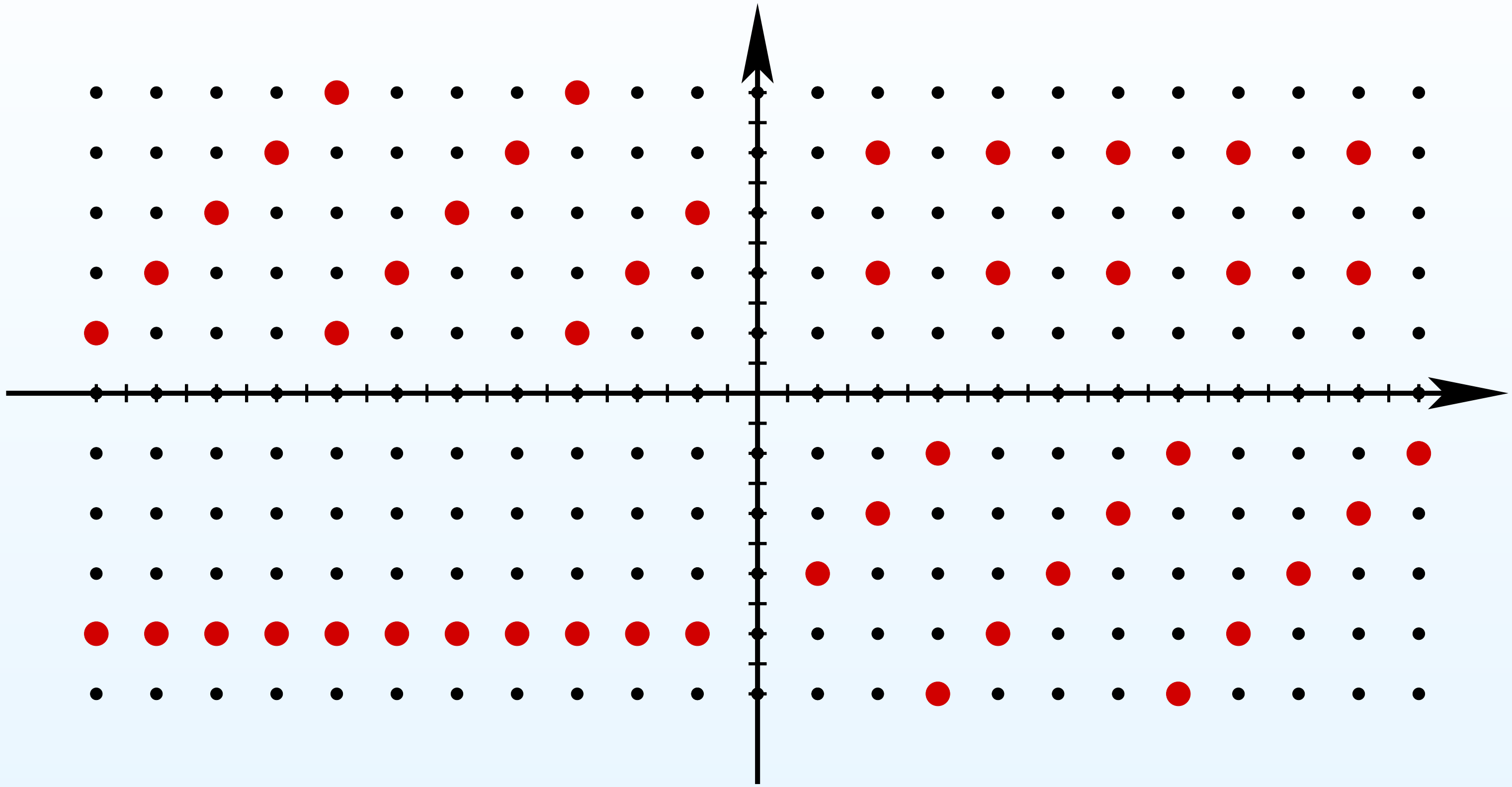


Finally, instead of using the entire lattice $\mathbb{Z}^n$ we can use any sublattice $\mathbb{L}^n \subset \mathbb{Z}^n$ having some nonzero *density* $k > 0$ in $\mathbb{Z}^n$.

For example, the set of coprime integral points in $\mathbb{Z}^2$ has density $k = \dfrac{6}{\pi^2}$ and can be also used to define the Lebesgue measure (scaled by the factor k) in any of the two ways discussed above.

Mirzakhani's measures on $\mathcal{ML}_{g,n}$



Choose some integral multicurve γ, say, a simple closed curve on $S_{g,n}$. The subset $\mathcal{O}_\gamma := \mathrm{Mod}_{g,n} \cdot \gamma$ can be seen as an analog of a "sublattice" in $\mathcal{ML}_{g,n}(\mathbb{Z})$. The insight of Mirzakhani was to realize that replacing the discrete set $\mathcal{ML}_{g,n}(\mathbb{Z})$ with the subset $\mathcal{O}_\gamma$ we get a new measure on $\mathcal{ML}_{g,n}$ which is proportional to the Thurston measure μ_{Th} with coefficient depending only on the homotopy type of γ.

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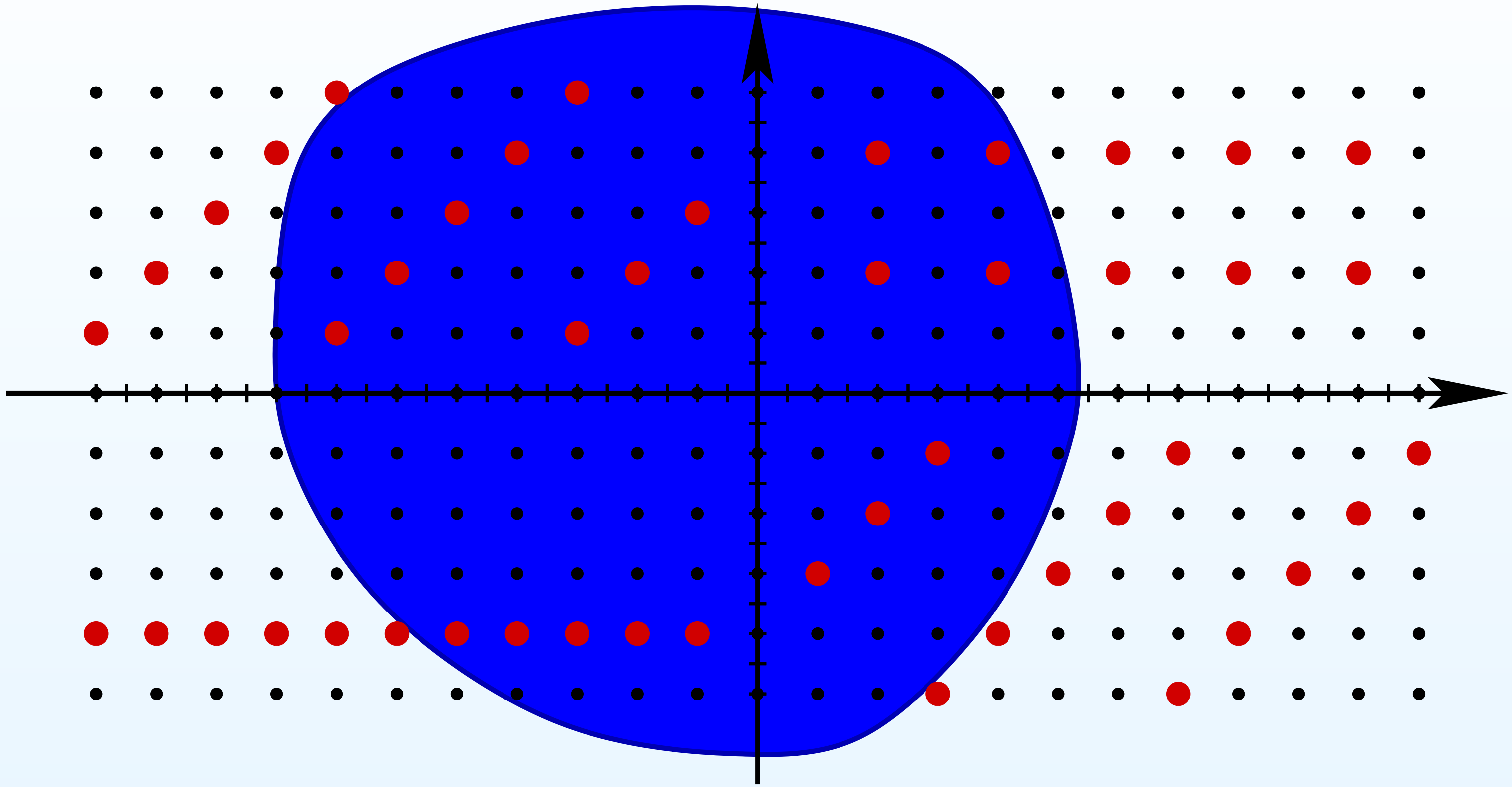

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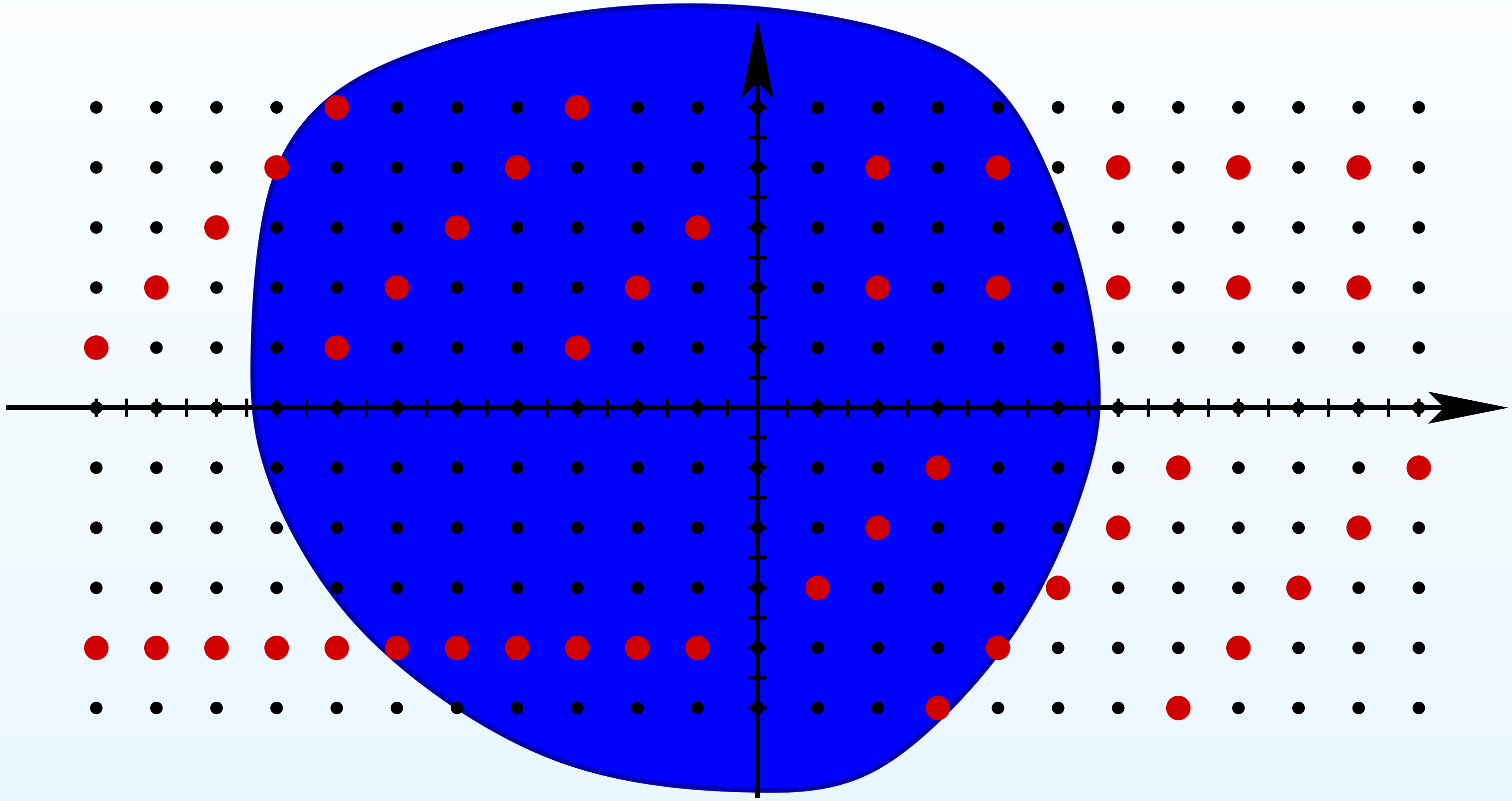

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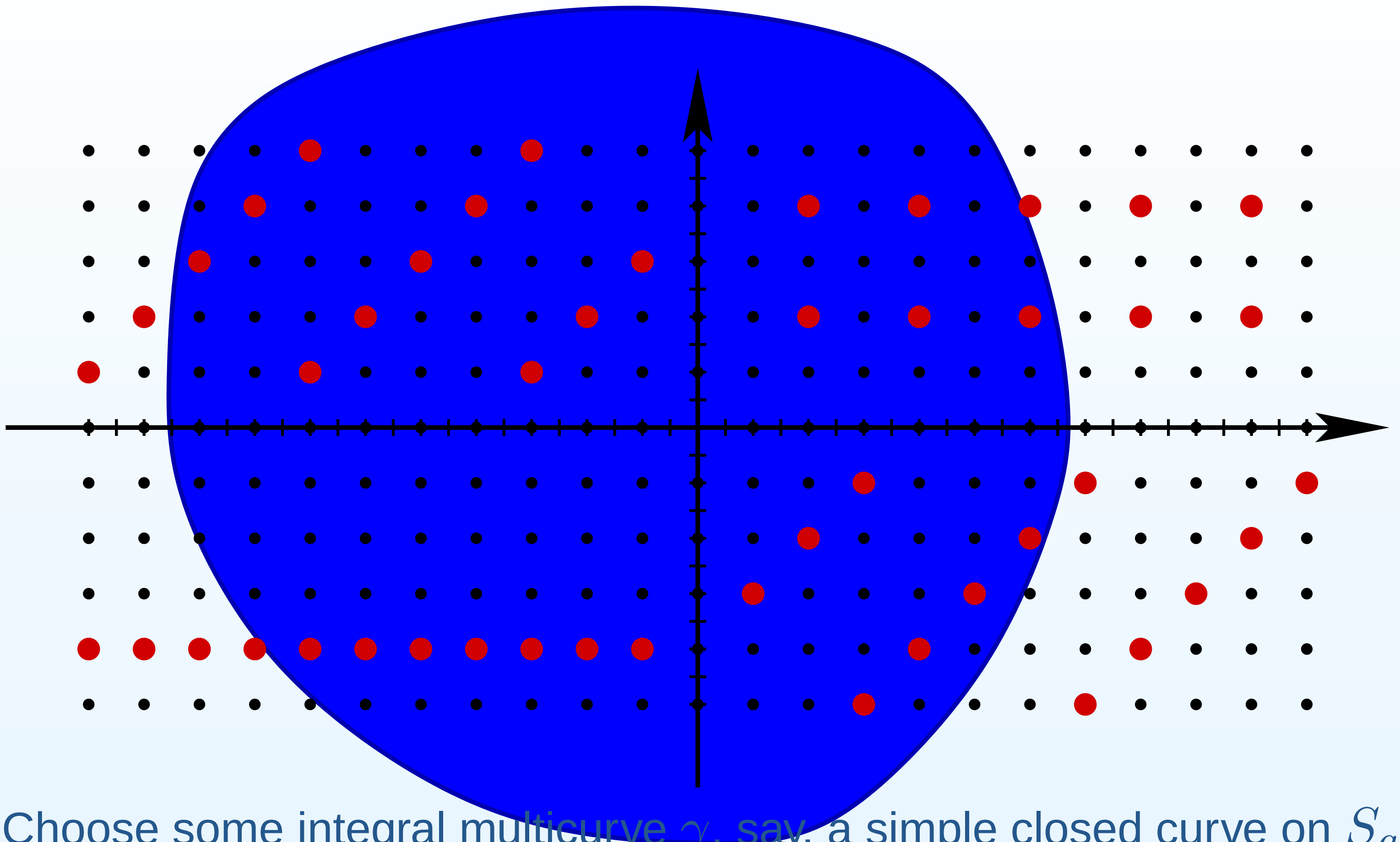

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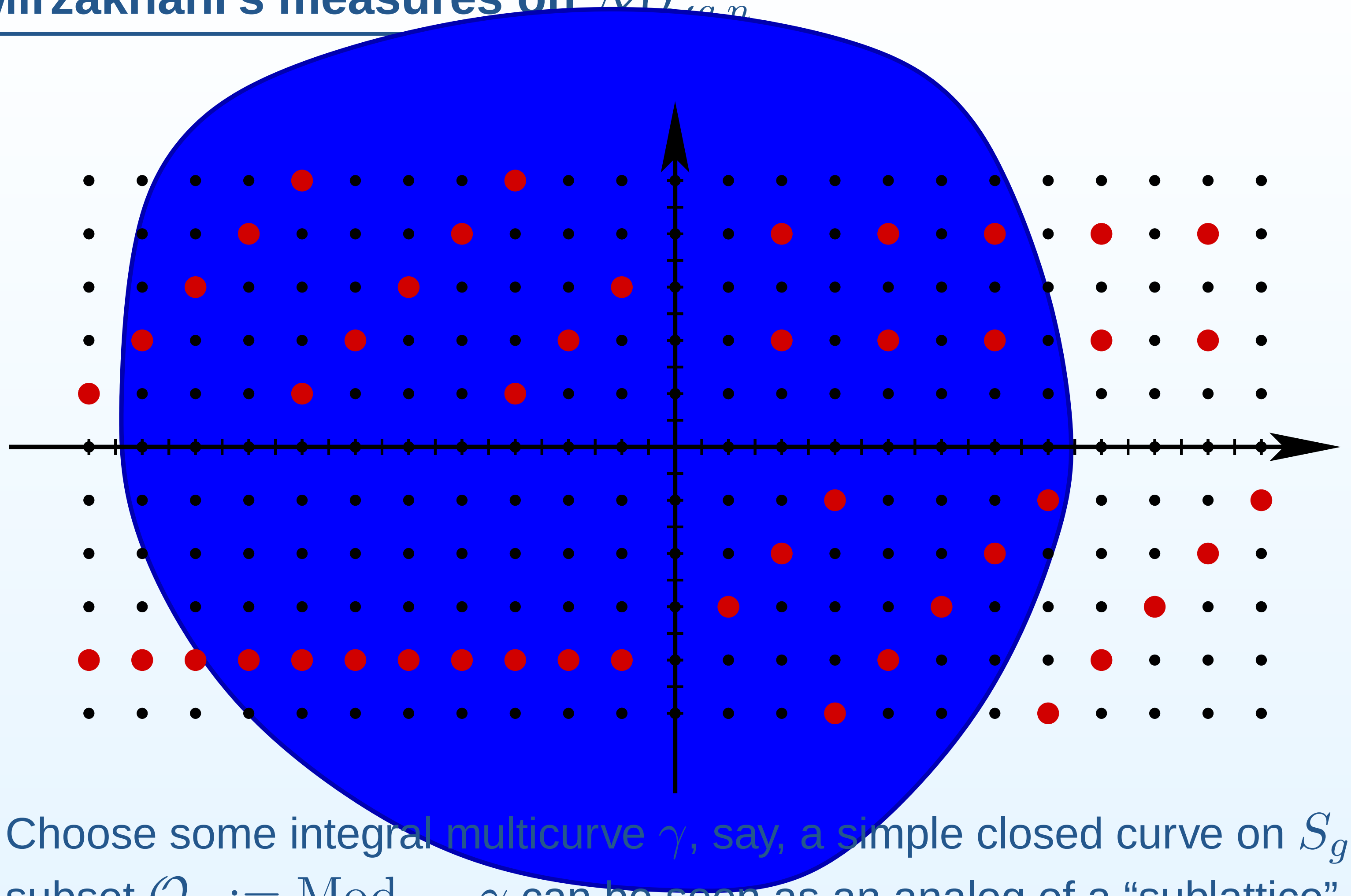

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Mirzakhani's measures on $\mathcal{ML}_{g,n}$

More formally: the Thurston measure of a subset $U \subset \mathcal{ML}_{g,n}$ is defined as

$$\mu_{\mathrm{Th}}(U) := \lim_{t \to +\infty} \frac{\operatorname{card}\{tU \cap \mathcal{ML}_{g,n}(\mathbb{Z})\}}{t^{6g-6+2n}} .$$

Mirzakhani defines a new measure μ_γ as

$$\mu_\gamma(U) := \lim_{t \to +\infty} \frac{\operatorname{card}\{tU \cap \mathcal{O}_\gamma\}}{t^{6g-6+2n}} .$$

Clearly, for any U we have $\mu_\gamma(U) \leq \mu_{\mathrm{Th}}(U)$ since $\mathcal{O}_\gamma \subset \mathcal{ML}_{g,n}(\mathbb{Z})$, so μ_γ belongs to the Lebesgue measure class. By construction μ_γ is $\mathrm{Mod}_{g,n}$-invariant. Ergodicity of μ_{Th} implies that $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ where $k_\gamma = const$.

Proof of the main result

Length function and unit ball

The hyperbolic length $\ell_\gamma(X)$ of a simple closed geodesic γ on a hyperbolic surface $X \in \mathcal{T}_{g,n}$ determines a real analytic function on the Teichmüller space. One can extend the length function to simple closed multicurves $\ell_{\sum a_i\gamma_i} = \sum a_i \ell_(\gamma_i)(X)$ by linearity. By homogeneity and continuity the length function can be further extended to $\mathcal{ML}_{g,n}$. By construction $\ell_{t\cdot\lambda}(X) = t \cdot \ell_\lambda(X)$.

Each hyperbolic metric X defines its own “unit ball” B_X in $\mathcal{ML}_{g,n}$:

$$B_X := \{\lambda \in \mathcal{ML}_{g,n} \,|\, \ell_\lambda(X) \leq 1\}\,.$$

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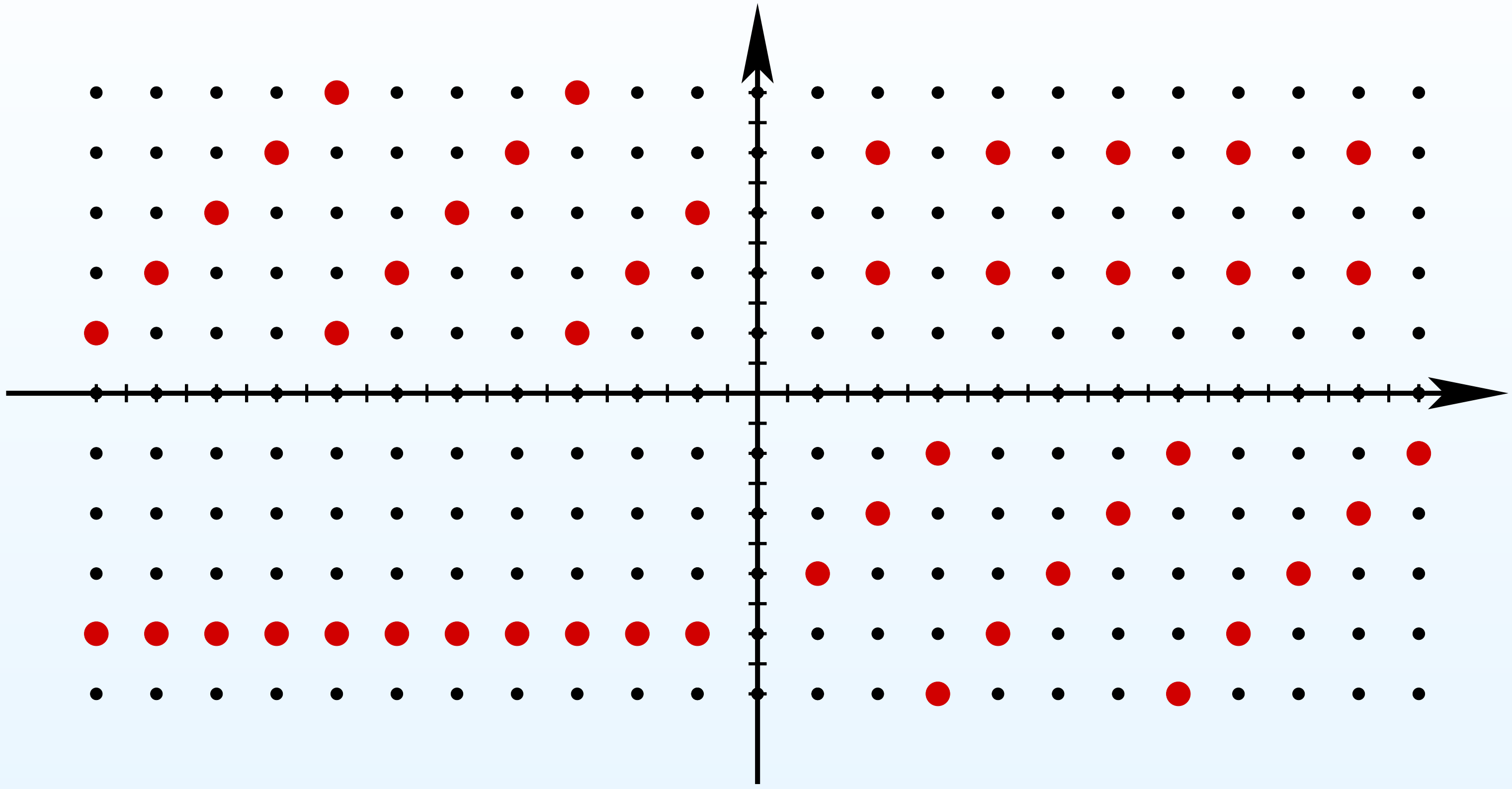

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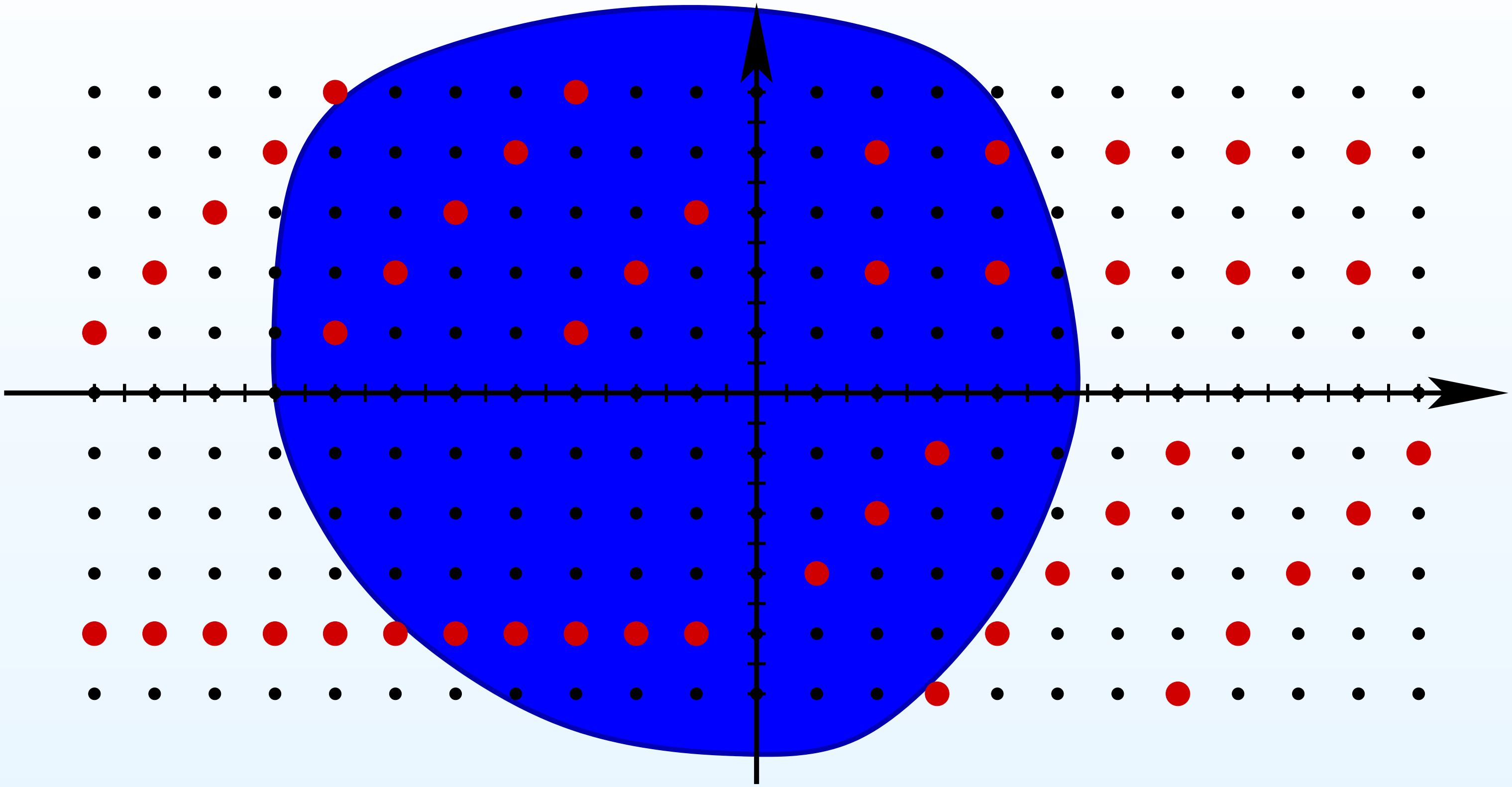

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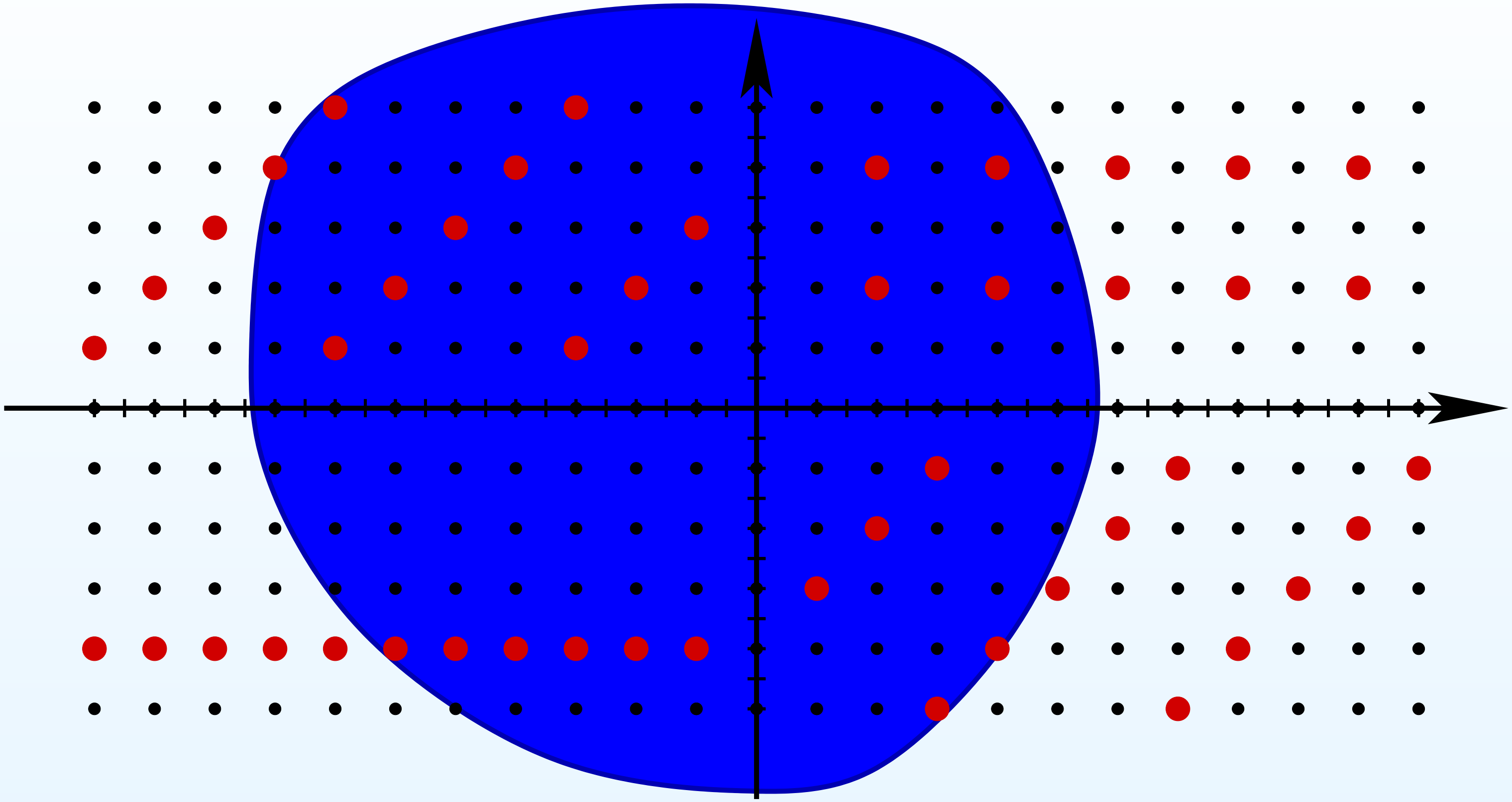

By definition of μ_{Th}, the Thurston volume of the unit ball is equal to the normalized number of integral points in a "ball of radius L" associated to X:

$$\mu_{\mathrm{Th}}(B_X) = \lim_{L \to +\infty} \frac{\mathrm{card}\{\lambda \in \mathcal{ML}_{g,n}(\mathbb{Z}) \mid \ell_\lambda(X) \leq L\}}{L^{6g-6+2n}} .$$

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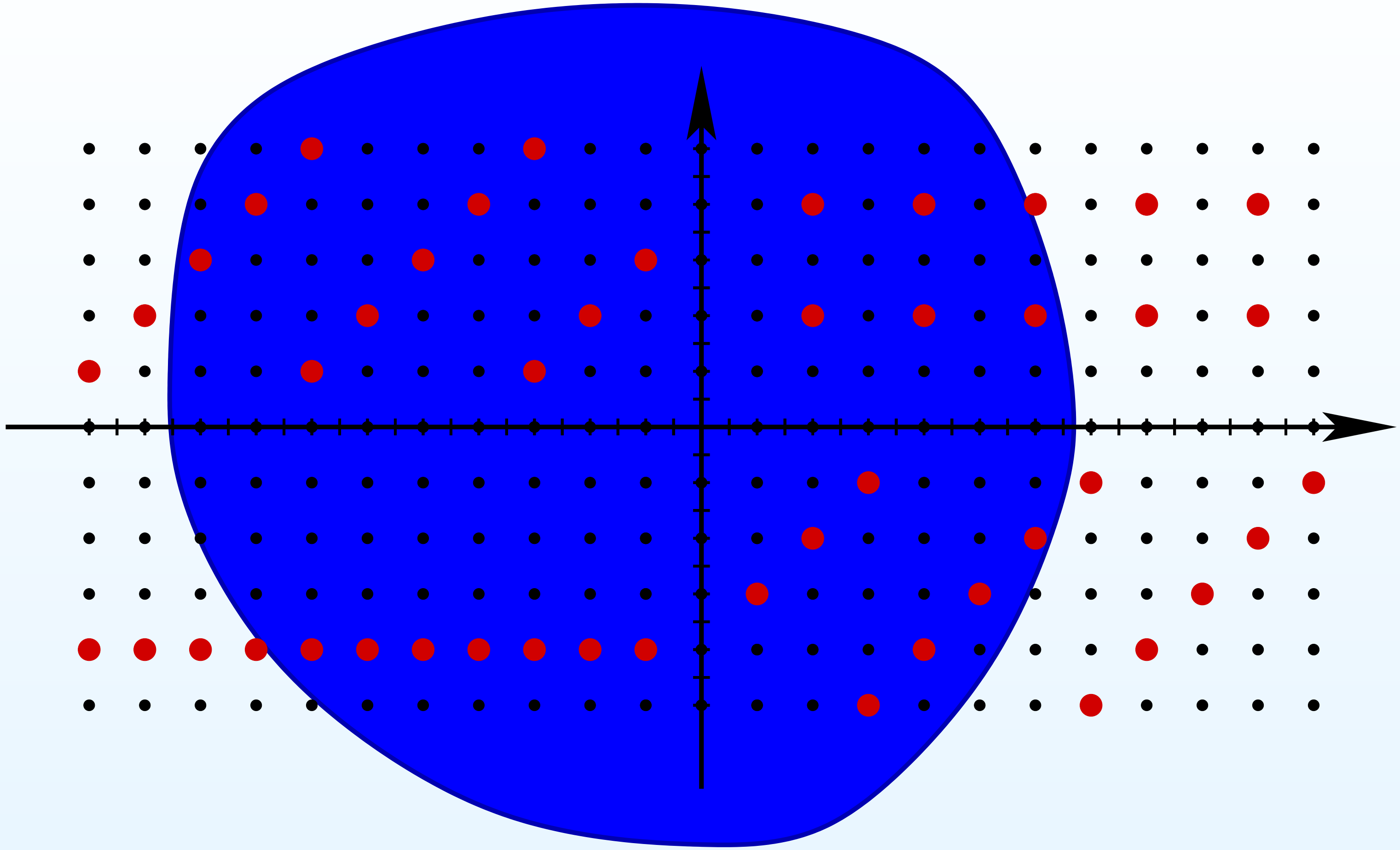

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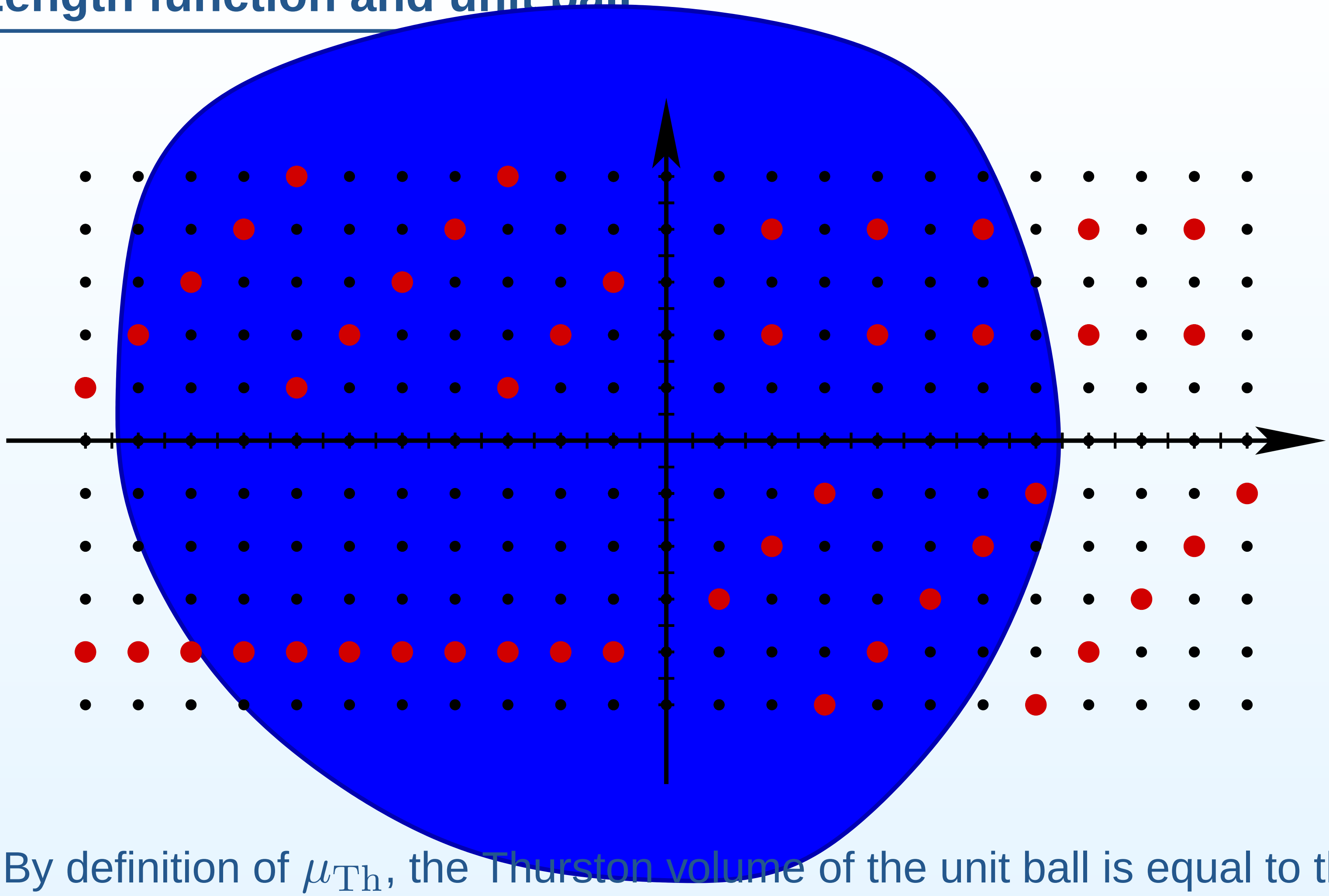

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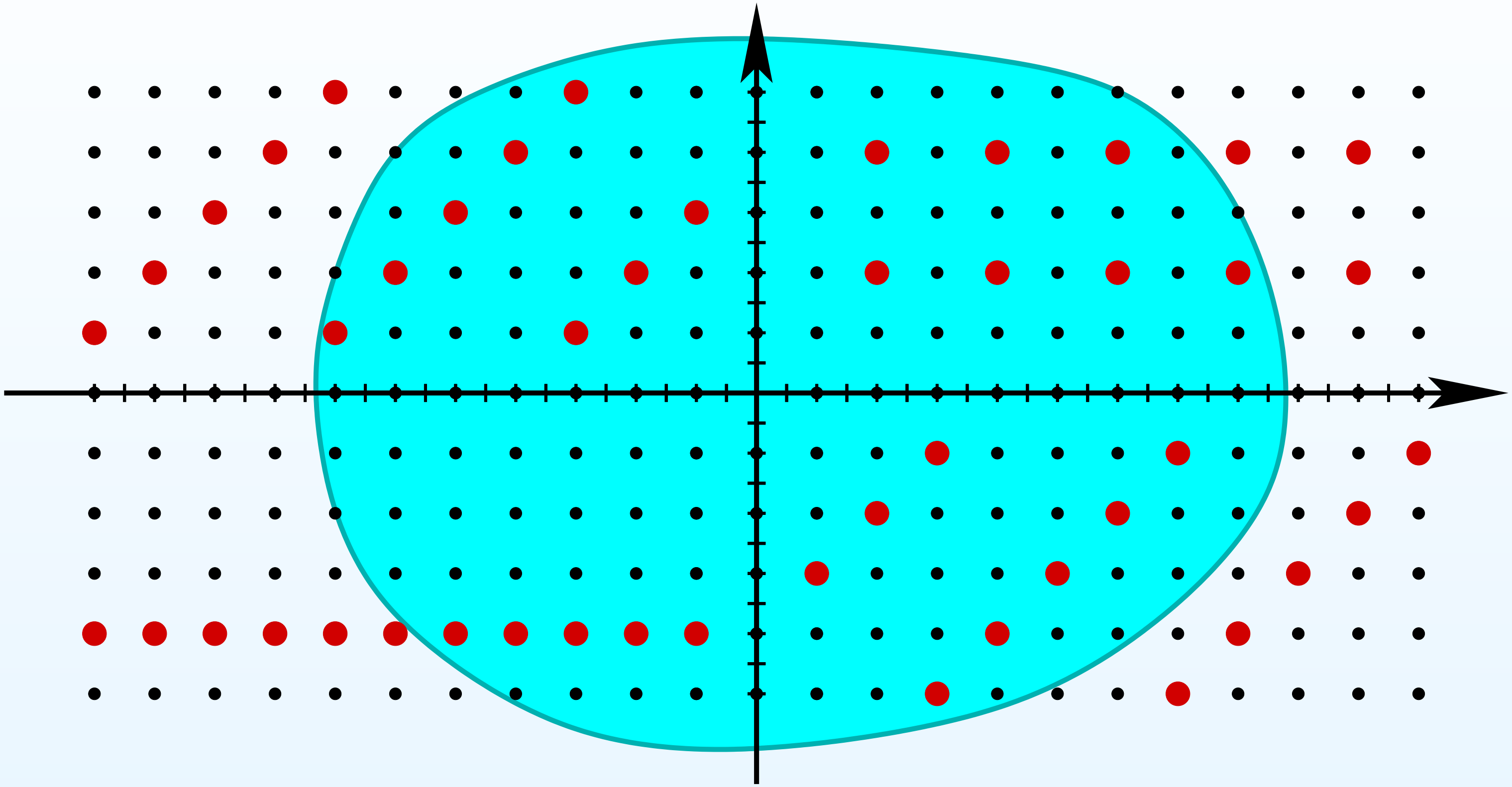

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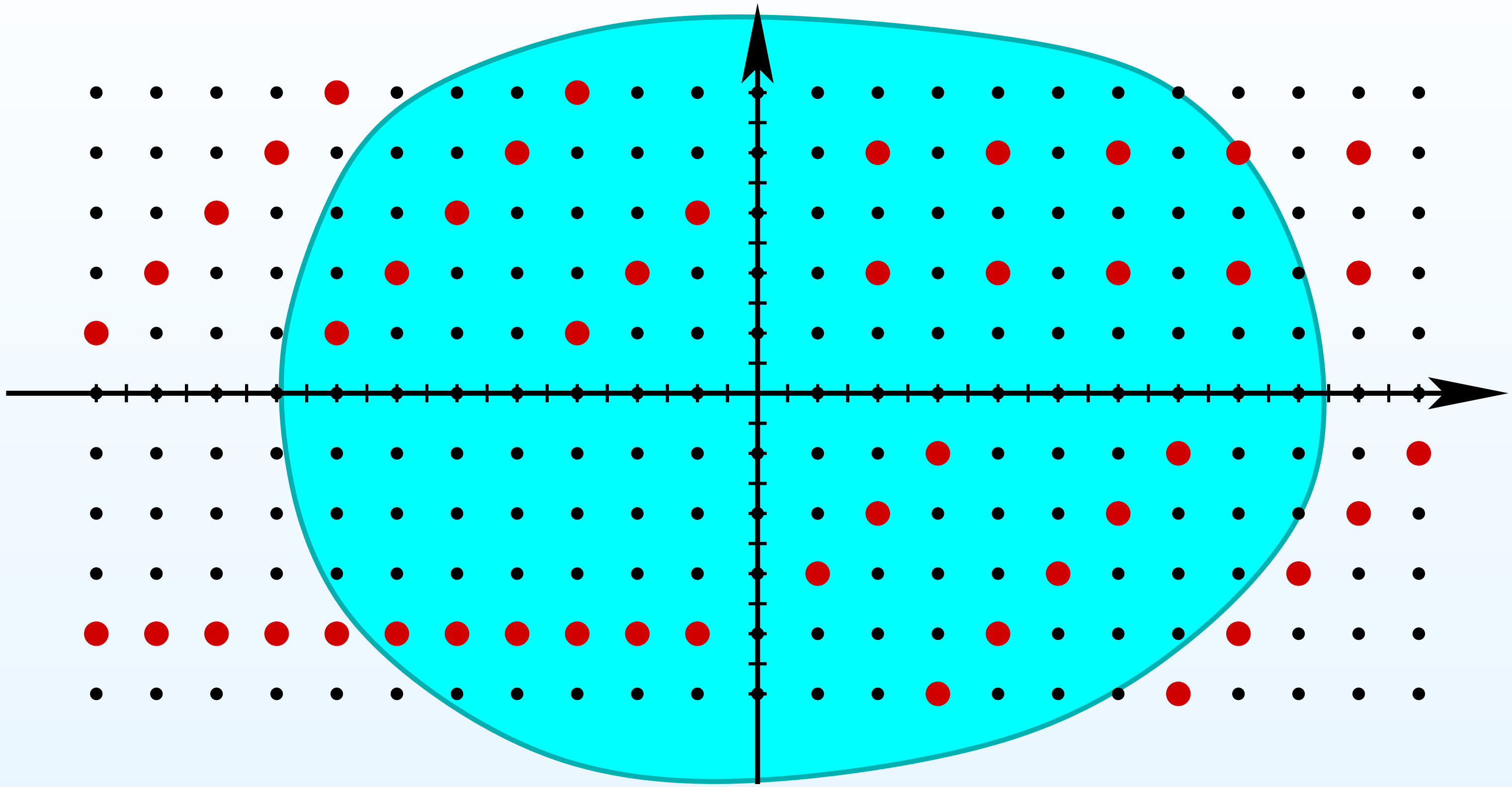

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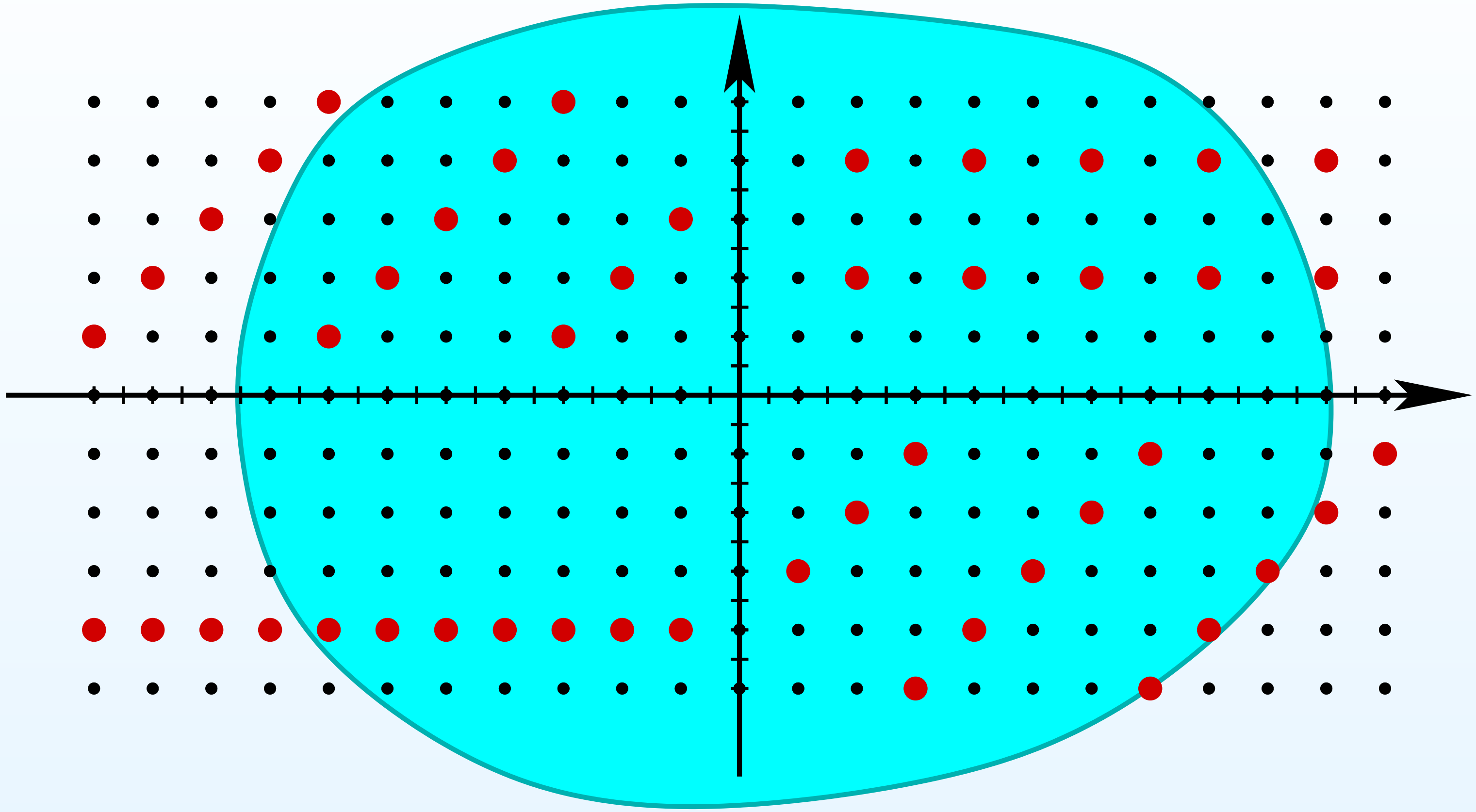

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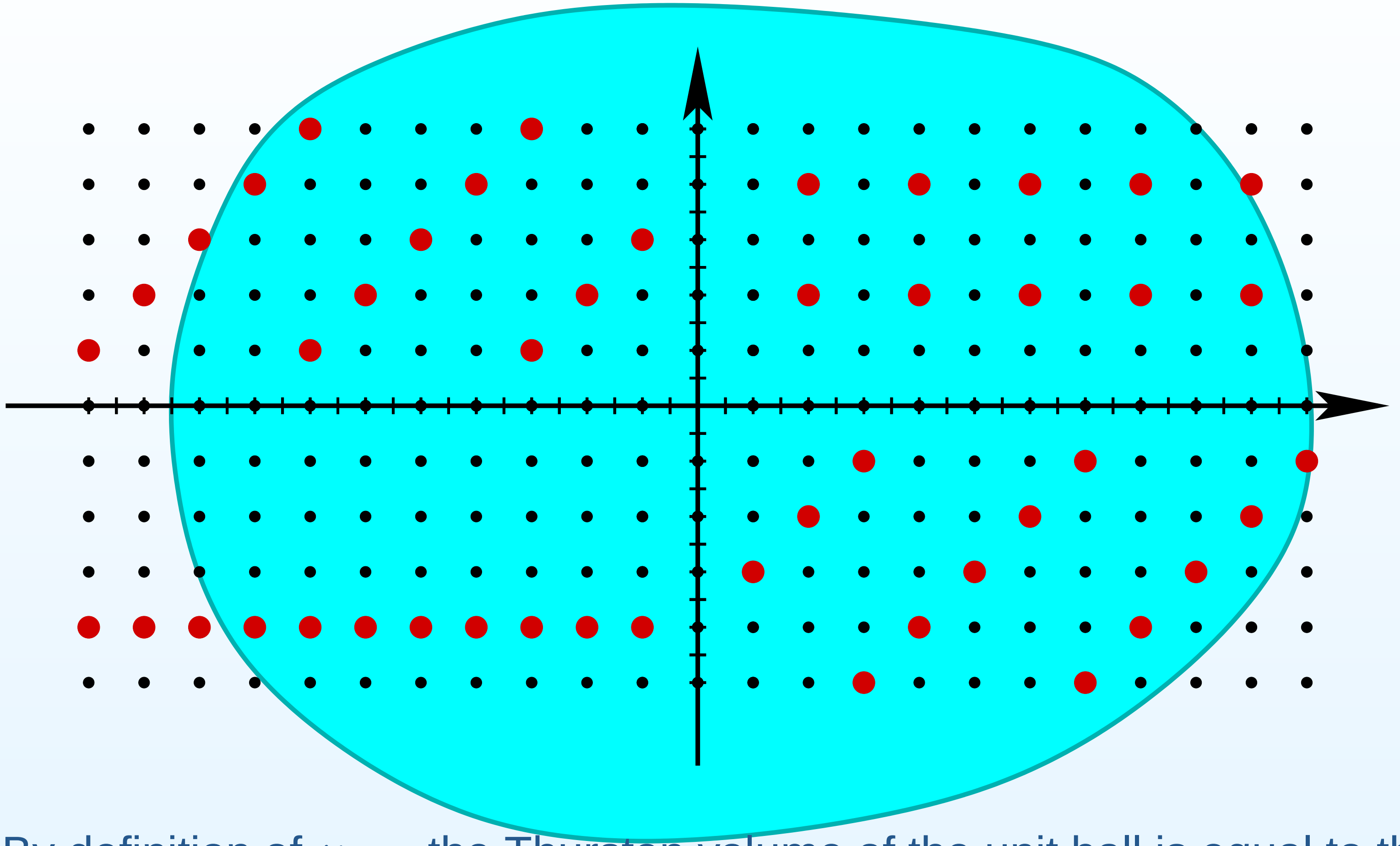

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Summary of notations

- X — a hyperbolic surface in $\mathcal{M}_{g,n}$.
- $s_X(L,\gamma)$ — the number of geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L.
- $P(L\,,\gamma) := \int_{\mathcal{M}_{g,n}} s_X(L,\gamma)\,dX$ — the polynomial in L providing the *average* number of geodesic multicurves of topological type $[\gamma]$ and of hyperbolic length at most L over all hyperbolic surfaces $X \in \mathcal{M}_{g,n}$.
- $c(\gamma)$ — the coefficient of the leading term $L^{6g-6+2n}$ of the polynomial $P(L\,,\gamma)$.

- $B(X)$ — "Unit ball" in $\mathcal{ML}_{g,n}$ defined by means of the length function $\ell_X(\alpha)$, where $\alpha \in \mathcal{ML}_{g,n}$.
- $\mu_{\mathrm{Th}}(B(X)) := \lim_{L\to+\infty} \dfrac{\operatorname{card}\{L\cdot B_X \cap \mathcal{ML}(\mathbb{Z})\}}{L^{6g-6+2n}}$ is the Thurston measure of the unit ball $B(X)$
- $\mu_\gamma(B(X)) := \lim_{L\to+\infty} \dfrac{\operatorname{card}\{L\cdot B_X \cap \mathrm{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}}$ is the Mirzakhani measure of the unit ball $B(X)$ defined by the sublattice $\mathrm{Mod}_{g,n}\cdot\gamma \subset \mathcal{ML}(\mathbb{Z})$.

Statement of the counting result

Theorem (M. Mirzakhani, 2008). *For any rational multi-curve* γ *and any hyperbolic surface* X *in* $\mathcal{M}_{g,n}$ *one has*

$$s_X(L,\gamma) \sim \mu_{\mathrm{Th}}(B(X)) \cdot \frac{c(\gamma)}{b_{g,n}} \cdot L^{6g-6+2n} \quad \text{as } L \to +\infty\,,$$

where

$$b_{g,n} := \int_{\mathcal{M}_{g,b}} \mu_{\mathrm{Th}}(B(X))\, dX$$

is the average Thurston measure of unit balls $B(X)$.

Completion of the proof

Recall that $s_X(L,\gamma)$ denotes the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L. Applying the definition of μ_γ to the "unit ball" B_X associated to hyperbolic metric X (instead of an abstract set B) and using proportionality of measures $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ we get

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = \lim_{L\to+\infty} \frac{\operatorname{card}\{L\cdot B_X \cap \operatorname{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}} = \mu_\gamma(B_X) = k_\gamma \cdot \mu_{\mathrm{Th}}(B_X)\,.$$

Finally, Mirzakhani computes the scaling factor k_γ as follows:

$$k_\gamma \cdot b_{g,n} = \int_{\mathcal{M}_{g,n}} k_\gamma \cdot \mu_{\mathrm{Th}}(B(X))\,dX = \int_{\mathcal{M}_{g,n}} \mu_\gamma(B(X))\,dX =$$

$$= \int_{\mathcal{M}_{g,n}} \lim_{L\to+\infty} \frac{\operatorname{card}\{L\cdot B_X \cap \operatorname{Mod}_{g,n}\cdot\gamma\}}{L^{6g-6+2n}}\,dX = \int_{\mathcal{M}_{g,n}} \lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}}\,dX =$$

$$= \lim_{L\to+\infty} \frac{1}{L^{6g-6+2n}} \int_{\mathcal{M}_{g,n}} s_X(L,\gamma)\,dX = \lim_{L\to+\infty} \frac{P(L,\gamma)}{L^{6g-6+2n}}\,dX = c(\gamma)\,,$$

so $k_\gamma = c(\gamma)/b_{g,n}$. Interchanging the integral and the limit we used the estimate of Mirzahani $\frac{s_X(L,\gamma)}{L^{6g-6+2n}} \le F(X)$, where F is integrable over $\mathcal{M}_{g,n}$.

Completion of the proof

Recall that $s_X(L,\gamma)$ denotes the number of simple closed geodesic multicurves on X of topological type $[\gamma]$ and of hyperbolic length at most L. Applying the definition of μ_γ to the "unit ball" B_X associated to hyperbolic metric X (instead of an abstract set B) and using proportionality of measures $\mu_\gamma = k_\gamma \cdot \mu_{\mathrm{Th}}$ we get

$$\lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}} = \lim_{L\to+\infty} \frac{\mathrm{card}\{L \cdot B_X \cap \mathrm{Mod}_{g,n} \cdot \gamma\}}{L^{6g-6+2n}} = \mu_\gamma(B_X) = k_\gamma \cdot \mu_{\mathrm{Th}}(B_X)\,.$$

Finally, Mirzakhani computes the scaling factor k_γ as follows:

$$k_\gamma \cdot b_{g,n} = \int_{\mathcal{M}_{g,n}} k_\gamma \cdot \mu_{\mathrm{Th}}(B(X))\,dX = \int_{\mathcal{M}_{g,n}} \mu_\gamma(B(X))\,dX =$$

$$= \int_{\mathcal{M}_{g,n}} \lim_{L\to+\infty} \frac{\mathrm{card}\{L \cdot B_X \cap \mathrm{Mod}_{g,n} \cdot \gamma\}}{L^{6g-6+2n}}\,dX = \int_{\mathcal{M}_{g,n}} \lim_{L\to+\infty} \frac{s_X(L,\gamma)}{L^{6g-6+2n}}\,dX =$$

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Average volume of unit balls

Recall that

$$b_{g,n} := \int_{\mathcal{M}_{g,b}} \mu_{\mathrm{Th}}(B(X))\, dX$$

denotes the average volume of “unit balls” measured in Thurston measure.

Theorem (M. Mirzakhani, 2008). *The quantity $b_{g,n}$ admits explicit expression as a weighted sum of all $c(\gamma)$ over (a finite collection) of all topological types $[\gamma]$ of multicurves.*

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Theorem (V. Delecroix, E. Goujard, P. Zograf, A. Z., 2017).

$$b_{g,n} = \mathrm{Vol}_{\mathrm{MV}}\, \mathcal{Q}_{g,n} = \mathrm{Vol}_{\mathrm{MV}}\, \mathcal{Q}(1^{4g-4+n}, -1^n)\,.$$

Mirzakhani's volume polynomials

Theorem (M. Mirzakhani, 2008). *Weil–Petersson volume of the moduli space of boarded hyperbolic surfaces is a polynomial in lengths of boundary components* $b_1^2, \dots, b_n^2$*. Its term of top degree* $3g-3+n$ *has the form:*

$$\mathrm{Vol}_{\mathrm{WP}}\left(\mathcal{M}_{g,n}\right)(b_1^2,\dots,b_n^2) = \frac{2}{2^{5g-6+2n}} \sum_{|d|=3g-3+n} \frac{\langle \psi_1^{d_1} \dots \psi_n^{d_n} \rangle}{d_1! \dots d_n!} b^{2d_1} \dots b^{2d_n} +$$

$$+(terms\ of\ lower\ degree).\ Here\quad \langle \psi_1^{d_1} \dots \psi_n^{d_n} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}\,.$$

Example: $\boldsymbol{\mathcal{M}_{1,1}}$. Here $3g-3+n=1$; $5g-6+2n=1$; $\langle\psi_1^1\rangle = \frac{1}{24}$, so

$$\mathrm{Vol}_{\mathrm{WP}}\left(\mathcal{M}_{1,1}\right)(b_1^2) = \frac{2}{2^1}\frac{\langle\psi_1\rangle}{1!} b_1^{2\cdot 1} + lower\ terms = \frac{1}{24} b_1^2 + lower\ terms\,.$$

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Here $3g-3+n=2$; $5g-6+2n=3$; $\langle\psi_1^2\rangle = \langle\psi_1\psi_2\rangle = \langle\psi_2^2\rangle = \frac{1}{24}$.

$$\begin{aligned}\mathrm{Vol}_{\mathrm{WP}}\left(\mathcal{M}_{1,2}\right)(b_1^2,b_2^2) &= \frac{2}{2^3}\left(\frac{\langle\psi_1^2\rangle}{2!\,0!} b_1^{2\cdot 2} + \frac{\langle\psi_1\psi_2\rangle}{1!\,1!} b_1^{2\cdot 1} b_2^{2\cdot 1} + \frac{\langle\psi_2^2\rangle}{0!\,2!} b_2^{2\cdot 2}\right) + \cdots \\ &= \frac{1}{192}(b_1^4 + 2b_1^2b_2^2 + b_2^4) + lower\ terms\,.\end{aligned}$$

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