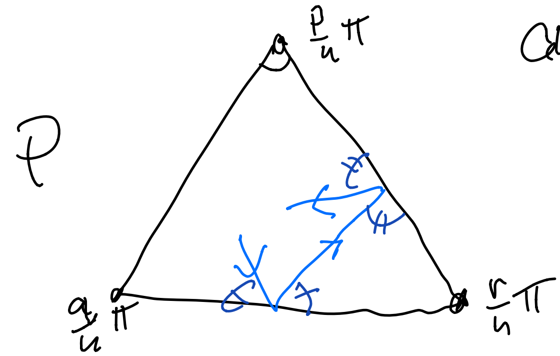
① Polygonal billiards.

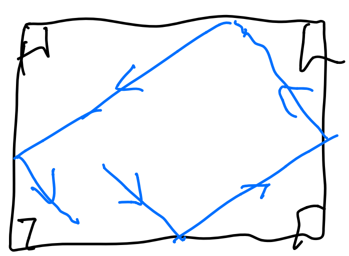
Billiard trajectories in polygons



angle of incidence = angle of reflection.

A polygon P is <u>rational</u> if all the angles are rational multiples of π.

Example 1: Classic billiard table



Questions: What is the long term behavior of trajectories? How does it depend on the polygon P? The direction θ? The initial point?

What sequences of sides is hit by a given trajectory?
(This question is related to the coding sequences discussed by Pascal & Sasha.)

What sequences of sides can be hit by some trajectory in some direction?
(This is important when you have to call your shot.)

For which θ's are all trajectories

dense?

uniformly distributed?

In which directions do we have periodic trajectories?

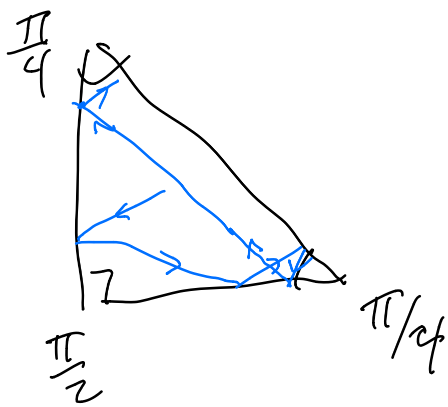
Do periodic trajectories always exist and if so how many are there?

In these 4 lectures we will consider the question of the asymptotic behavior of the number of trajectories of length at most L.

<u>Connection between rational billiards and translation surfaces</u>

In a rational polygon P each billiard trajectory travels in only finitely many directions.

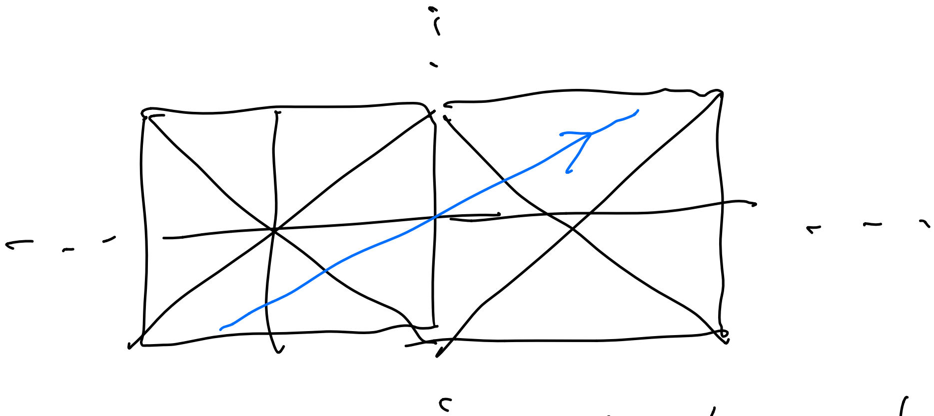
Example 2:



8 directions

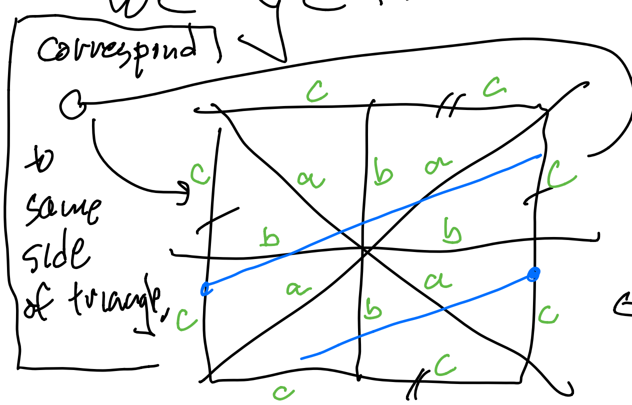
Unfolding construction converts billiard trajectories to straight lines - Euclidean geodesics.

In this case unfolding the triangle gives a tiling of the plane.



We can thus relate billiard trajectories to straight lines in $\mathbb{R}^2$.

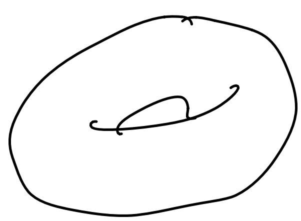
If we only unfold 8 times we get:



We have exactly one triangle for each possible direction of a fixed trajectory.

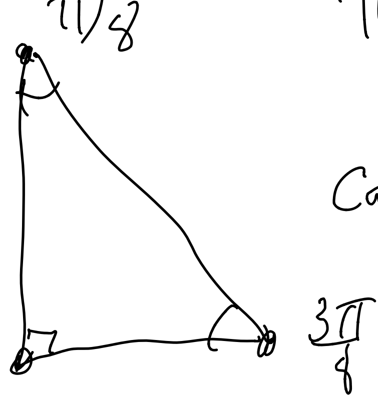
fundamental domain for a lattice acting by translations

Identify opposite sides by translations:

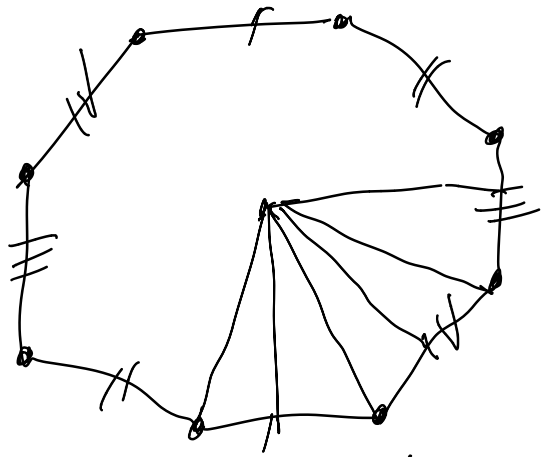


This construction gives a useful relation between billiard traj and geodesics in $T^2 = \mathbb{R}^2 / \Lambda$.

Example 3



This triangle does not tile the plane. Can see that by unfolding around $\frac{3\pi}{8}$ vertex. However there is still an analog of the previous torus construction. Octagon built from 16 triangles.



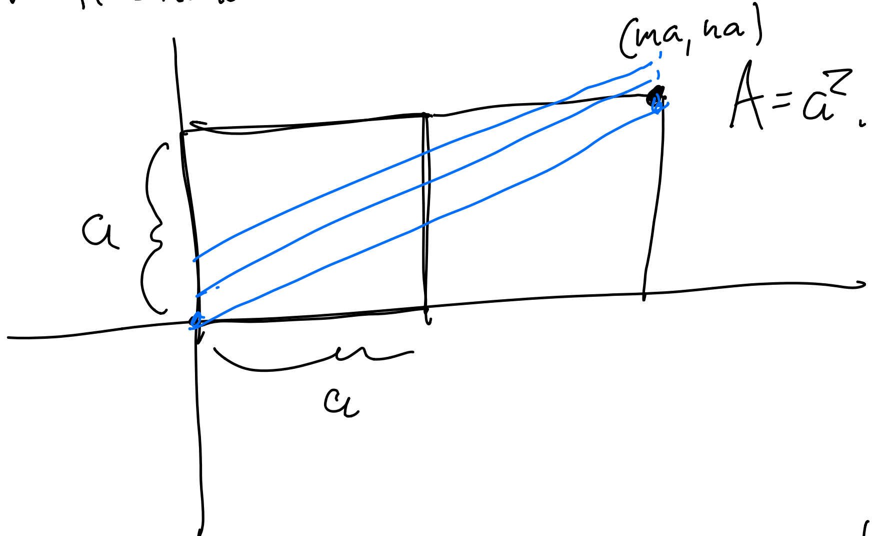
In this case we get a translation surface of genus 2 with a sing. pt. of cone angle 6π. (Check this!)

If P is a rational polygon we can associate associate a translation surface M_P. This construction was described by Zemlyakov-Katok and earlier by Fox-Kershner. Let Γ be the subgroup of $O(2)$ generated by reflections in the sides of P. For each $\gamma \in \Gamma$ construct a disjoint polygon $\gamma P \in \mathbb{R}^2$. Glue appropriate parallel sides together.

Let's return to the counting problem.

Example: Torus of area A

or a polygon that tiles the plane by reflection



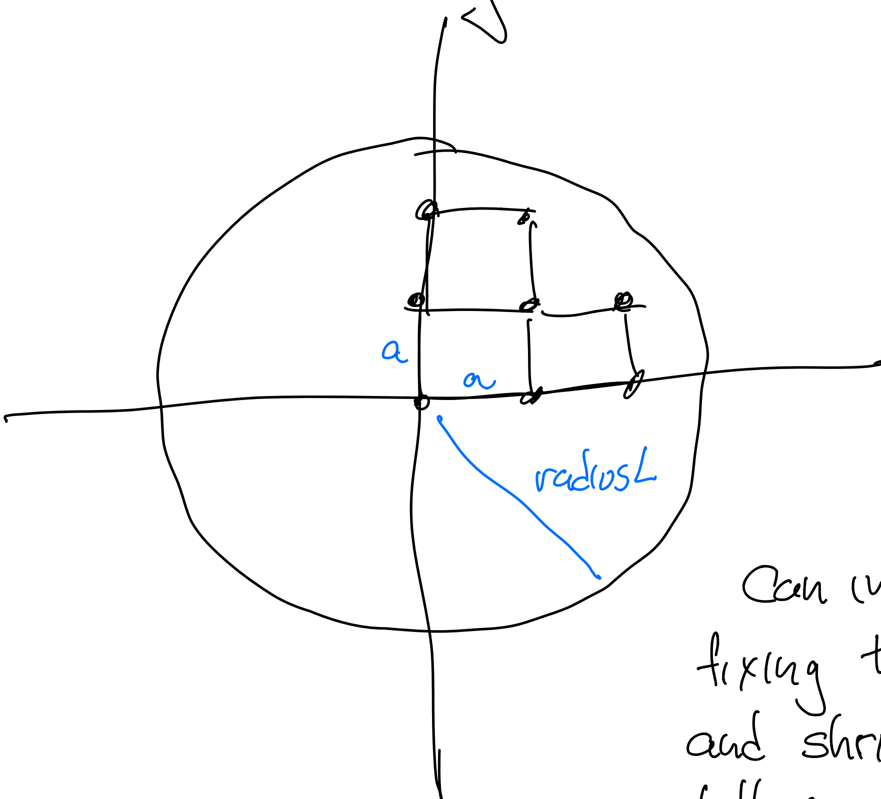
$A = a^2.$

A closed geodesic corresponds to a point $\begin{pmatrix} ma \\ na \end{pmatrix}$ with $m, n \in \mathbb{Z}$.

Conversely such a pair gives

rise to a family of closed geodesics all of the same length. $\ell = \sqrt{m^2a^2 + n^2a^2} = a\sqrt{m^2+n^2}$.

How many such families are there of length $\leq L$?



Can imagine fixing the disk and shrinking the lattice.

$$\# \approx \frac{\pi L^2}{A}.$$

This overcounts by considering the same geod. multiple times. To correct the count we consider only pairs $\binom{m}{n}$ which are relatively prime.

We are also considering each geod. twice due to orientation.

Chance that m, n rel. prime

$= \frac{1}{\zeta(2)}$ where $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

Get $N_p(L) \sim \frac{\pi L^2}{A} \cdot \frac{1}{2} \cdot \frac{1}{\zeta(2)}$

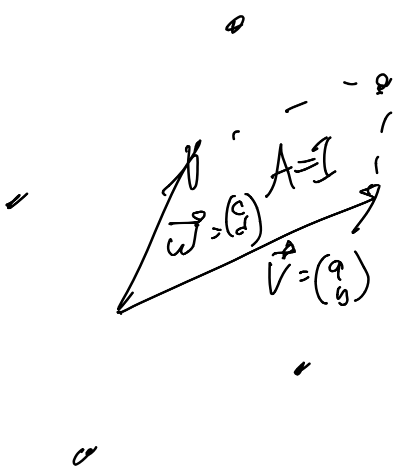
$$= c \cdot \frac{L^2}{A} \qquad c = \frac{\pi}{2 \cdot \zeta(2)}$$

Problem with this technique is that it relies on the structure of $\mathbb{R}^2$ or $\tilde{M}$. For the general rational polygon this is not available.

Let's use a "renormalization technique" in the case of the torus and see what we get.

(stratum) We can identify the "moduli space" of tori ($\mathcal{H}(0)$) of area 1 with the space of unimodular lattices.

Recall that the space of unimodular marked lattices can be identified with $SL(2,\mathbb{R})$.



$$\sim \begin{pmatrix} \vec{v} & \vec{w} \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2,\mathbb{R}).$$

Changing the basis to a new oriented basis corresponds to the right action of $SL(2,\mathbb{Z})$ on $SL(2,\mathbb{R})$

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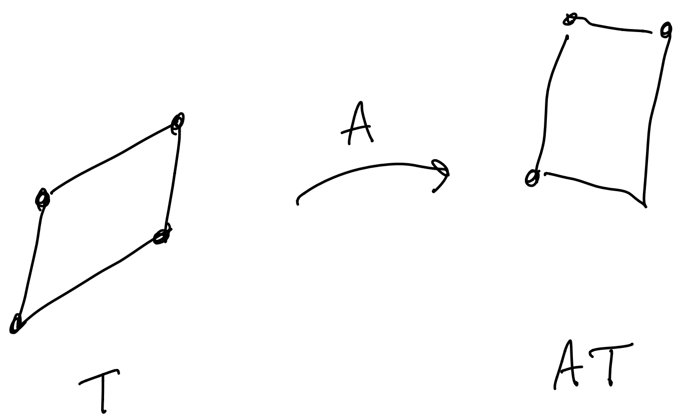
stratum of tori with 1 marked point:

$$\mathcal{H}_1(0) = SL(2,\mathbb{R}) / SL(2,\mathbb{Z}).$$

Deforming the lattice by a linear transf. corresponds to left mult by an element of $SL(2,\mathbb{R})$.



We call this the geometric action of $SL(2,\mathbb{R})$ on the space of tori. (14/10)

Recall that Pascal considered the one parameter family

$g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ (?) (Conformal distortion e^t)

Chris considered $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ in defining Teichmüller distance. (conformal distortion e^{2t}).

First normalization is more closely related to hyperbolic geometry and we will use that. (Gives geod. flow param. at unit speed.)

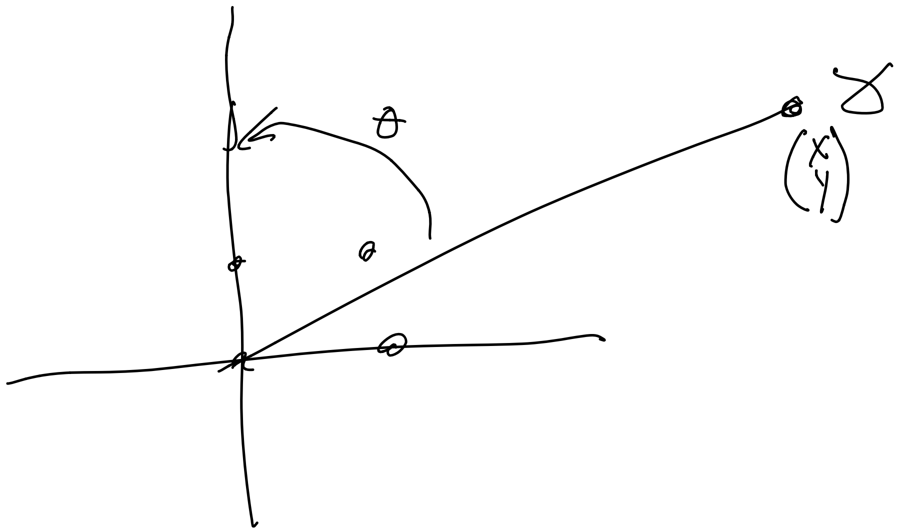
We can think of g_t as rescaling the vertical flow on $T = \mathbb{R}^2/\Lambda$.

The problem with counting geodesics in general is that geodesics are long.

We want to use g_t to make closed geodesics shorter.

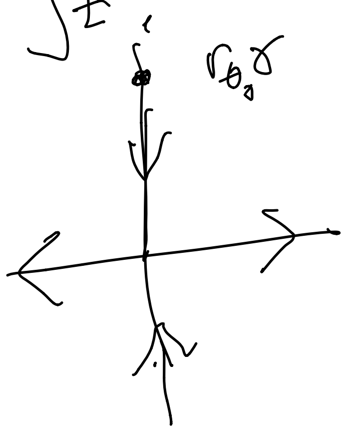
Let $r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

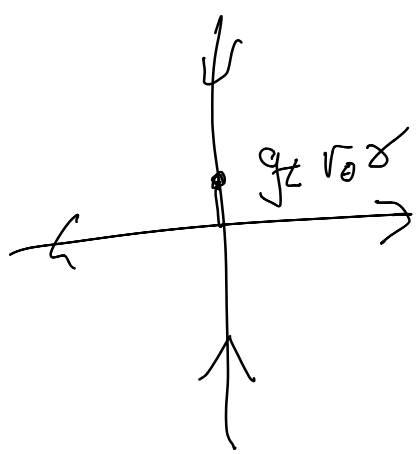
Let $\gamma = \begin{pmatrix} am \\ an \end{pmatrix}$ be a geodesic of length L_γ. Pick $\varepsilon > 0$. Let's deform the torus $T = \mathbb{R}^2/\Lambda$ to create a new torus $T' = \mathbb{R}^2/\Lambda'$ in which γ has length ε.



Rotate by r_{θ_γ} and apply

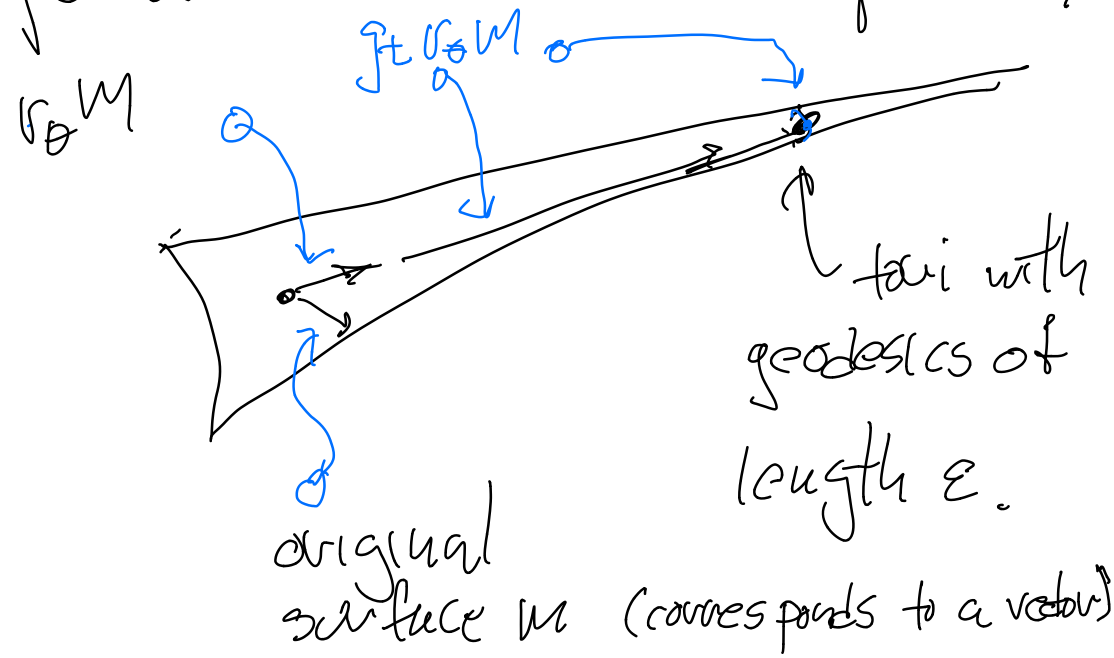
g_t,





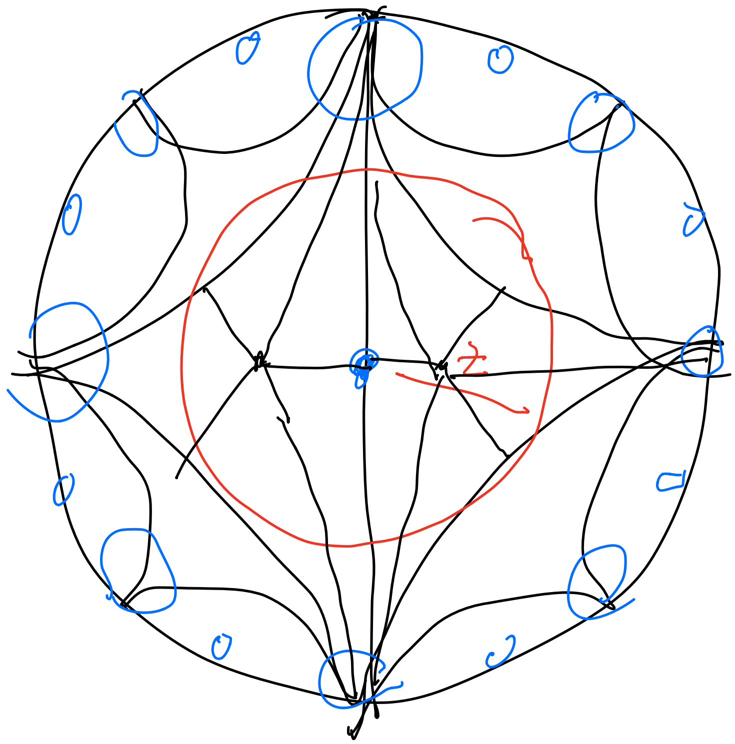
In order to do this we choose $t = 2\log\left(\frac{L}{\varepsilon}\right)$.

What is the effect of this operation on moduli-space?

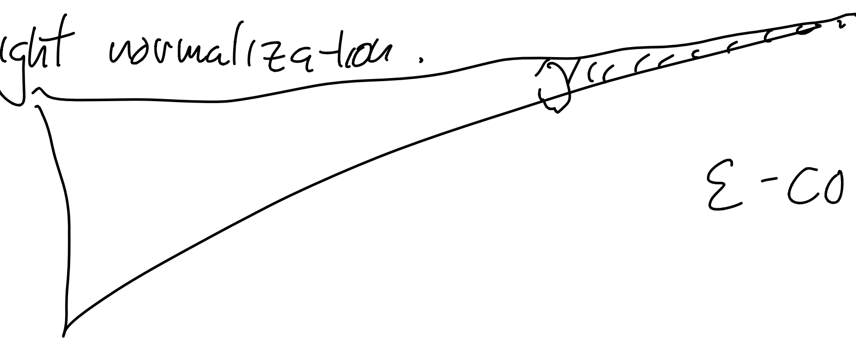


Let C_ε be the set of tori with geodesics of length $\le \varepsilon$.

Alternate picture:



The set $\{g_t r_\theta : \theta \in [0, 2\pi)\}$ corresponds to the circle of radius t since we chose the right normalization.



ε-cusp

Think of ripples in a pond spreading out from a point.

Geodesics of length less than L correspond to visits of the circle of radius t to the ε-cusp.

Quadratic growth comes from the fact that the hyperbolic length of this circle is

$$2\pi\sinh(t) \approx 2\pi \cdot \frac{e^t}{2}$$

$$= \pi \cdot e^{2\log(4/\varepsilon)}$$

$$= \frac{\pi}{\varepsilon} \cdot L^2 \qquad \text{as } L \to \infty.$$

The assertion that circles become uniformly distributed as $L \to \infty$ implies that the proportion of time that the circle spends in the cusp is:

$$\frac{\mathrm{vol}(C_\varepsilon)}{\mathrm{vol}(SL(2,\mathbb{R})/SL(2,\mathbb{Z}))}$$

as $L \to \infty$.

Assume that each excursion to the cusp intersects the circle in an interval of a fixed length: l_ε

Assuming that each excursion to the cusp corresponds to a θ interval of l_ε implies that the number of geodesics is

$$\frac{\pi}{\varepsilon} \cdot L^2 \cdot \frac{\mathrm{vol}(C_\varepsilon)}{\mathrm{vol}\left(SL(2,\mathbb{R}) / SL(2,\mathbb{Z}) \right)} \cdot \frac{1}{\ell_\varepsilon}$$

More careful analysis shows that the answer is independent of ε.

Remark: Can calculate these vols. using Gauss-Bonnet and conclude that $\zeta(2) = \frac{\pi^2}{6}$.

How do we apply these ideas to other polygons?

Start with a translation surface M.

$\mathcal{H}(0) = SL(2,\mathbb{R})/SL(2,\mathbb{Z})$

M is in a "moduli space" $\mathcal{H}(\alpha)$.

We have some analog of the ε-cusp, $C_\varepsilon \subseteq \mathcal{H}(\alpha)$.

We want to know that circles become uniformly distributed in $\mathcal{H}(\alpha)$ with respect to Haar measure?

In fact circles are all contained in $SL(2,\mathbb{R})\cdot M \subset \mathcal{H}(\alpha)$ so we want to apply these ideas to $\overline{SL(2,\mathbb{R})\cdot M} \subset \mathcal{H}(\alpha)$.

Are these orbit closures nice submanifolds? Do they have $SL(2,\mathbb{R})$ invariant measures which are easy to calculate with?

Thm. (Eskin-Mirzakhani-Mohammadi) Yes, $SL(2,\mathbb{R})$ orbit closures in strata have all the properties you need to make these arguments work.

More specifically: $SL(2,\mathbb{R})$ orbit closures are submanifolds and have unique smooth $SL(2,\mathbb{R})$ invariant probability measures.

Question to discuss in next 3 lectures:

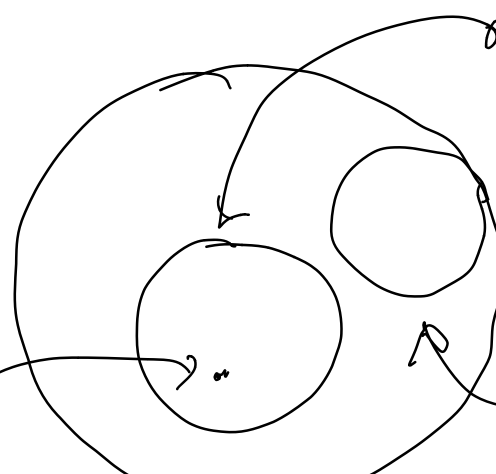
Problem. Do large circles equidistribute in orbit closures?

If the answer is yes then we can attack the problem by finding orbit closures and computing the appropriate constants.

Large circles in the hyperbolic plane:

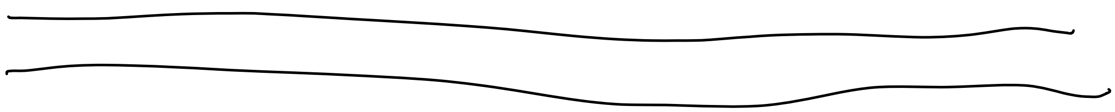
Unlike large Euclidean circles the curvature of large hyperbolic circles tends to 1 rather than 0. Curves of curvature 1 are called horocycles.

Key to analyzing limits of large circle measures is analyzing the horocycle flow.



Curvature of the circle of radius r

is $\tanh r = \dfrac{e^t - e^t}{e^t + e^{-t}} \to 1$ as $t \to \infty$.



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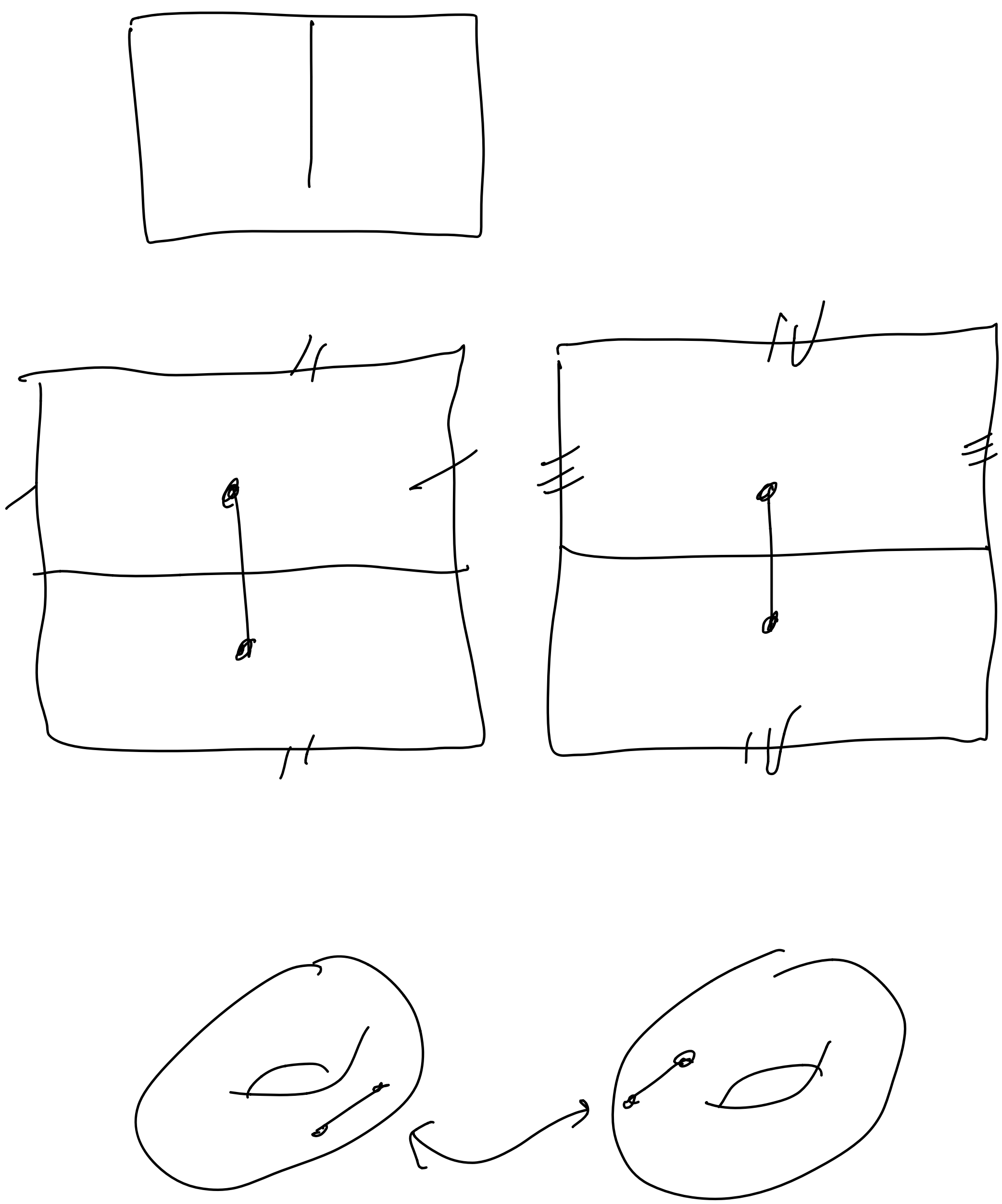

Consider the set of surfaces M with maps $\pi: M \to T$ which branch over 2 points.

Call the set of such surfaces $\mathcal{E}_4$.

The set of such surfaces is locally determined by 5 parameters: 3 describe the shape of the torus and 2 describe the relative positions of the branch points.