[1958c] Final report on contract AF 18(603)-57

Within the framework of the project originally submitted to AFOSR, I eventually decided to concentrate on two lines of investigation:

(I) The classical problem of the moduli of algebraic curves over complex numbers;

(II) A study of the Kähler varieties topologically identical with the non-singular quartics in projective 3-space (henceforward called K3 surfaces).

In both directions, my results are still very fragmentary and incomplete; and I have had to postpone the arithmetical considerations which provided the original motivation for the whole project, in order to deal first with the function-theoretic aspects of the above questions.

In both problems, the ideas of Kodaira and Spencer on the variation of complex structures have proved fundamental. I have much benefited from repeated consultation with them during my stays in Princeton, in January and February and again in June. It also turned out that Professor L. Bers had been engaged in a parallel investigation of problem (I); consultation with him on this topic has been very fruitful. On the other hand, various aspects of problem (II) have recently engaged the attention of Professor L. Nirenberg, Professor A. Andreotti, and Dr. Atiyah; I have learned a great deal from communications, written and oral, from all of them.

In order to give, in what follows, a coherent account of these topics, it will be necessary to include much of the work of my colleagues, and it would be unpractical to try to unravel in detail what may belong to me and what belongs to each one of them. It should be understood that they deserve a large share of the credit for the work described in this report.

I. *Moduli of algebraic curves*

We consider curves of a given genus $g \geq 1$. One basic concept is that of a Teichmüller structure. If S, S' are two oriented surfaces of genus g, we say that a class of mappings of S into S' in the sense of homotopy (or, briefly, a class $C(S, S')$) is admissible if it contains at least one orientation-preserving homeomorphism of S onto S'. Let S_0 be an oriented surface of genus g, given once for all. By a Teichmüller surface, we understand a Riemann surface of genus g (i.e. a surface of genus g, provided with a complex-analytic structure and oriented accordingly), together with an admissible class $C(S_0, S)$. Isomorphism being defined for these in the obvious manner, we introduce, with Teichmüller, a space T (the "Teichmüller space"), whose points correspond in one-to-one manner to all the classes of mutually isomorphic Teichmüller surfaces of genus g. Teichmüller's chief contribution was

to define on T a certain topology, the "natural" one in a sense described below, and then to prove that T, with this topology, is homeomorphic to an open cell of real dimension $6g - 6$. So far, I have mainly been concerned with the local properties of the Teichmüller space and of its "natural" complex-analytic structure.

The definition of the latter depends upon ideas introduced by Teichmüller himself, but which do not appear to have been fully understood until Kodaira and Spencer attacked similar problems for higher dimensions. In order to describe them, it is convenient to substitute for the above definition of a Teichmüller surface the following one. Let $A_1, \ldots, A_{2g}$ be a set of generators, fixed once for all, of the fundamental group G_0 of S_0 (with a given origin), satisfying the relation

$$R(A_1, \ldots, A_{2g}) = A_1 A_2 A_1^{-1} A_2^{-1} \cdots A_{2g}^{-1} = 1.$$

A set $\sigma_1, \ldots, \sigma_{2g}$ of linear-fractional substitutions on z will be called admissible if it has the following properties: (a) it generates a discrete group G of hyperbolic substitutions acting on P; (b) $S = P/G$ is a compact surface of genus g; (c) there is an admissible class $C(S_0, S)$, mapping G_0 onto G considered as the fundamental group of S, which maps A_i onto σ_i for $1 \leq i \leq 2g$. Each Teichmüller surface S has a universal covering which can be mapped conformally onto P, and can therefore be represented as P/G; the class $C(S_0, S)$ which belongs to it defines an isomorphism of G_0 onto G; therefore, to each Teichmüller surface, there corresponds at least one admissible set $(\sigma_1, \ldots, \sigma_{2g})$. Conversely, it is easy to see that two such sets will define isomorphic Teichmüller surfaces if and only if they can be transformed into one another by a linear-fractional substitution. Using this fact, it is possible to normalize admissible sets in such a way that there is a one-to-one correspondence between classes of Teichmüller surfaces (i.e. points of the space T) and normalized admissible sets (σ_i); this gives a one-to-one mapping of T onto a subset of the coordinate space R^{6g-6}. It turns out that the latter subset is open, and that the mapping and its inverse are both indefinitely differentiable (and even, presumably, real-analytic) if T is provided with its "natural" differentiable structure; the proof of the latter fact is due to L. Bers.

Now introduce a "variation of structure" as follows. Let μ be any indefinitely differentiable complex-valued function in P such that $|\mu| < 1$; let P_μ denote the upper half-plane with the modified complex structure for which $dz + \mu\, d\bar{z}$ is a differential form of type (1, 0); if the latter structure is invariant under G, i.e. if $\mu\, d\bar{z}/dz$ is formally invariant under G in an obvious sense, the complex structure of P_μ can be projected onto a complex structure P_μ/G, making the latter into a Riemann surface S_μ, or rather a Teichmüller surface if we keep the σ_i as the distinguished generators of G. Let F_μ, in that case, be the conformal mapping of P onto P_μ; the Teichmüller surface S_μ is then the one defined by the admissible set $\sigma_i(\mu) = F_\mu^{-1}\sigma_i F_\mu$; S_μ is isomorphic to S if and only if F_μ (which is defined only up to a fractional-linear substitution) can be chosen so that it commutes with all the σ_i; in that case, the "variation of structure" defined by μ will be called trivial.

From this, we get an "infinitesimal variation" if we take μ to depend (differentiably) upon a real parameter t, so that $\mu = 0$ for $t = 0$; then $\nu = (d\mu/dt)_{t=0}$ is called an infinitesimal variation of structure; it is clear that any function ν in P, such that $\nu\, d\bar{z}/dz$ is formally invariant under G, defines such a variation. The variation ν will be called trivial if the finite variation μ is tangent, for $t = 0$, to a trivial one, i.e. if there is a trivial variation μ', depending upon t, such that $d\mu/dt = d\mu'/dt$ for $t = 0$. It is easily seen that a necessary and sufficient condition for this is that there should exist a function ξ in P such that $\nu = \partial\xi/\partial\bar{z}$ and that ξ/dz is formally invariant.

The infinitesimal variations ν can be considered as the elements of a vector-space V (of infinite dimension) over the complex numbers; the trivial variations make up a subspace V' of V. Standard procedures in cohomology theory and the theory of fibre-bundles (which, in a case like this one, depend merely upon elementary facts such as Stokes's theorem and the theorem of Riemann-Roch) show that V/V' is of finite dimension $3g - 3$ and can be "canonically" identified with the dual space to the space of quadratic differentials of the first kind on S. As to the latter assertion, let $\omega = q\, dz^2$ be such a quadratic differential; in other words, we take for q a holomorphic function in P, such that $q\, dz^2$ is formally invariant under G. Then, for ν in V, $\nu q\, dz\, d\bar{z}$ is formally invariant under G, so that we may integrate it over P/G; Stokes's theorem shows that the integral is 0 for ν in V', i.e. it depends only upon the class D of ν modulo V'; denoting it by (ω, D), one finds that this bilinear form defines a duality between V/V' and the space of quadratic differentials, as asserted above.

At this point, one must make use of the fact (proved by Bers) that, if μ depends differentiably upon some real parameters, the same will be true of the mapping function F_μ and hence also of the coefficients in the substitutions $\sigma_i(\mu)$. It is now easy to calculate the effect of an infinitesimal variation on the coefficients of the σ_i, i.e. to calculate $d\sigma_i(\mu)/dt$ for $t = 0$, in terms of $\nu = (d\mu/dt)_{t=0}$. One can also calculate the effect of a given infinitesimal variation on the periods of the normalized integrals of the first kind on S. The conclusions one can derive from this are as follows. It is possible to provide T with a complex-analytic structure, of complex dimension $3g - 3$, such that, whenever μ depends holomorphically upon some complex parameters w_i, the point of T which corresponds to S_μ depends holomorphically upon the w_i; this observation is due to Bers, who also found, more precisely, that, if we take $\mu = \sum w_i y^2 \bar{q}_i$, with $y = \mathrm{Im}(z)$ and $(q_1, \ldots, q_{3g-3})$ such that $q_i\, dz^2$ are a basis of the space of quadratic differentials of the first kind on S, then the w_i can be taken as local complex coordinates in T in a neighborhood of the point corresponding to S. Furthermore, the quadratic differentials of the first kind on S can be identified with the covectors on T at that point; and Petersson's hermitian metric, in the space of those differentials (which is no other than the space of automorphic forms of degree -4 for the group G) defines an intrinsic Hermitian metric on T, which turns out to be a Kähler metric. The facts concerning the mapping of T into R^{6g-6} by means of the coefficients of the σ_i have already been stated. Finally, the periods of the normalized integrals of the first kind on S define a mapping of T into the Siegel space of symmetric $g \times g$ matrices with positive-definite imaginary part; the image of T under that mapping is a complex-analytic variety W, whose singular points are those corresponding to hyperelliptic Riemann surfaces; and one obtains a new proof for Rauch's theorem, stating which of the periods of the normalized integrals of the first kind can be used as local coordinates in the neighborhood of any given simple point of W.

II. The K3 surfaces

We may start here from the observation (made independently, I believe, by Atiyah and myself) that, when a non-singular surface S in projective 3-space acquires a node, i.e. a conical double point, and the latter is desingularized by a standard dilatation, this process gives a surface S' which is homeomorphic to S. It was easy to surmise that the same is true when a surface acquires any number of distinct nodes; this, in fact, or rather a much more precise theorem, was proved by Atiyah. It shows, in particular, that the non-singular quartic in 3-space, the double plane with a non-singular sextic branch curve, and the desingularized Kummer surface, are all homeomorphic.

Such surfaces will be called K3; they had already occurred in the work of the Italian geometers, and, more recently, in that of Kodaira. The Italians, in fact, had discovered an infinite sequence of families F_n $(n = 1, 2, \ldots)$ of regular surfaces with $p_g = 1$; F_1 consists of double planes with a sextic branch curve; F_2, of quartics in 3-space; F_n consists of surfaces of degree $2n$ in projective $(n + 1)$-space, whose hyperplane sections are canonical curves of genus $n + 1$. There are very plausible arguments to indicate that all such surfaces are of type K3, although no complete proof for this seems to have been given yet.

For K3 surfaces, the intersection matrix of two-dimensional cycles has the signature (19, 3), the determinant -1, and is even (i.e. the self-intersection of every cycle is even); hence there is exactly one double differential of the first kind; if we call it η, then we must have $d\eta = 0, \eta^2 = 0$ and $\eta\bar{\eta} \geq 0$; if we assume (as seems very likely) that all K3 varieties (algebraic or not) constitute only one connected family, then the canonical class must be 0, so that $\eta \neq 0$ and $\eta\bar{\eta} > 0$ everywhere; this implies that the complex structure is entirely determined by η.

Conversely, let there be given, on a differentiable manifold of that nature, a complex-valued differential form η of degree 2, satisfying $d\eta = 0$, $\eta^2 = 0$, and $\eta\bar{\eta} > 0$ everywhere; this determines a complex structure. It seems very plausible (but not at all easy to prove) that two such forms with the same periods must determine complex structures which can be transformed into one another by a differentiable homeomorphism, homotopic to the identity; that all such structures are Kählerian; and that the periods of η do not have to satisfy any other condition than those which are implicit in the relations $\eta^2 = 0, \eta\bar{\eta} > 0$.

These conjectures (which have also been made independently by Andreotti) may also be expressed as follows. Let S be a class of such structures, two structures being put into the same class if and only if they can be transformed into each other by a differentiable homeomorphism, homotopic to the identity. Let $(a_1, \ldots, a_{22})$ be a minimal set of generators for the two-dimensional homology group with integral coefficients; if η is as described above, let the p_i be its periods corresponding to the cycles a_i; let P be the point with the homogeneous coordinates $(p_1, \ldots, p_{22})$ in the complex projective space of dimension 21. Let $F(x, y)$ be the symmetric bilinear forms in $x = (x_1, \ldots, x_{22})$, $y = (y_1, \ldots, y_{22})$ whose matrix is the intersection-matrix of the cycles a_i. Then the conditions $\eta^2 = 0, \eta\bar{\eta} > 0$ imply that P is in the open subset H of the quadric $F(x, x) = 0$ which is determined by the inequality $F(x, \bar{x}) > 0$; H is a homogeneous space of complex dimension 20 for the orthogonal group determined by the real form $F(x, x)$. Now, if we assign to each class S of structures of the given type the point $P \in H$, we have a mapping of the set of all such classes into H; and Nirenberg, by an argument combining the Kodaira–Spencer technique with techniques derived from the theory of elliptic equations, has proved that the image of that set in H must be open. The conjectures stated above would mean that the mapping is a one-to-one mapping of that set onto H.

Furthermore, by a fundamental theorem of Kodaira, a given K3 structure will define an algebraic variety if and only if it has a Kähler metric whose fundamental form has integral periods, i.e. belongs to an integral cycle a; we must have $F(a, a) < 0$. For such a structure, the point P defined above must belong to the linear variety L_a defined by $F(a, x) = 0$, and therefore to the set $H_a = H \cap L_a$. It is easy to see that for each integral a such that $F(a, a) < 0$, H_a can be identified with the Riemannian symmetric space belonging to the orthogonal group of the quadratic form F_a of signature $(19, 2)$ induced by F on L_a. Again, one is led to conjecture that this establishes a one-to-one correspondence between such structures and H_a.

Now it may happen that two classes S, S' may be distinct and still define isomorphic structures; this will be so when structures belonging to these classes can be transformed into one another by a differentiable homeomorphism, not homotopic to the identity. The latter will induce an automorphism of the homology group, and therefore a unit U of the quadratic form $F(x, x)$, i.e. a matrix with integral coefficients belonging to the orthogonal group of F; let G be the group of all the units of F which can be obtained in this manner. Assuming the truth of the conjectures stated above, we see that two points P, P' of H will determine isomorphic structures if and only if they are equivalent under the group G. A similar statement will hold for H_a; here G has to be replaced by the subgroup G_a of G consisting of the elements which leave a invariant.

This shows that it is important to determine G, and in particular to determine whether G coincides with the group $\bar{G}$ of all the units of F. My results on this, too, are still incomplete. However, by applying the theory of theta functions to the Kummer surface considered as a model for K3 surfaces, it has been possible to reduce part of the problem to a purely arithmetical question which has been recently solved by M. Kneser. The result is that G is, at any rate, of finite index in $\bar{G}$.

Now, since H is not the Riemannian symmetric space for the orthogonal group of F, it follows that G does not act upon H in a properly discontinuous manner; hence there can be no theory of the moduli, in the ordinary sense, for the postulated connected family of K3 surfaces. This is as expected, and is analogous to the well-known fact that there is no theory of moduli for complex toruses, but only for "polarized" abelian varieties. The situation is quite similar here. For, if one restricts oneself to the family of algebraic K3 surfaces polarized by assigning a Kähler form belonging to a given integral cycle a, then it follows from what we have said that G_a acts in a properly discontinuous manner on H_a, and is of finite index in the group of all units of the quadratic form F_a, so that H_a/G_a is of finite measure. It is therefore to be expected that the automorphic functions in H_a, for the group G_a, make up an algebraic function-field, the field of the moduli for the K3 surfaces of the given family. One interesting feature here is the occurrence, in a problem of moduli, of the automorphic functions belonging to the group of units of a quadratic form of signature $(n, 2)$ (with $n = 19$ in the present case). This is believed to be

the first time that such a group has appeared in such context. Of course, before this can be more thoroughly investigated, it will be necessary to obtain full proofs for the conjectures stated above. After that is done, analogies with the theory of abelian varieties and of their fields of moduli (given by Siegel's modular functions) will undoubtedly suggest a number of further problems, of a function-theoretic and also of a number-theoretic nature; most fascinating, perhaps, are the possibilities suggested by the theory of complex multiplication. But this is still too remote to be discussed here.