Grenoble Summer School June 20 2018

More exercises.

1. Let ν_t be the measure on a stratum $\mathcal{H}$ defined as follows. For $M \in \mathcal{H}$, and $f \in C_c(\mathcal{H})$,

$$\int_{\mathcal{H}} f \, d\nu_t = \frac{1}{2\pi} \int_0^{2\pi} f(g_t r_\theta M) \, d\theta.$$

Suppose $t_i \to \infty$ and $\nu_{t_i} \xrightarrow{i \to \infty} \mu$ in weak-* topology, i.e. for all $f \in C_c(\mathcal{H})$,

$$\lim_{i \to \infty} \int_{\mathcal{H}} f \, d\nu_{t_i} = \int_{\mathcal{H}} f \, d\mu.$$

Prove that μ is U-inv.

2. Let $\mathcal{E}_4 = \{ M \in \mathcal{H}(1,1) : M$ is a 2:1 translation cover of a surface in $\mathcal{H}(0,0) \}$

and let $P : \mathcal{E}_4 \longrightarrow \mathcal{H}(0,0)$ be the map sending $M \in \mathcal{E}_4$ to the surface in $\mathcal{H}(0,0)$ which it covers.

(a) Determine the degree of P.

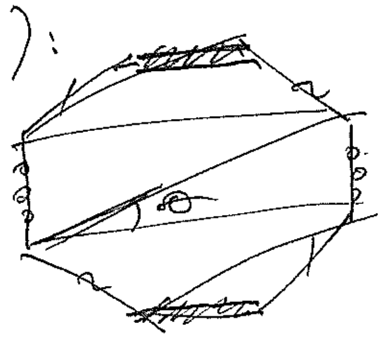
(b) Does P have branched points, i.e. are there $x \in \mathcal{H}(0,0)$ for which
$$\# P^{-1}(x) < \max_{y \in \mathcal{H}(0,0)} \# P^{-1}(y) ?$$

3. Suppose M is a union of horizontal cylinders $C_1, \dots, C_m$, with circumfrences c_i and heights h_i. Let $\mu_i = \frac{c_i}{h_i}$, and suppose there are $k_i \in \mathbb{N}$ s.t.

(*) $\quad k_1 \mu_1 = \dots = k_m \mu_m =: \mu$

Show $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in \Gamma_M$.

4. Let M be a regular $2n$-gon with opposite sides identified (see picture for $n=4$):



For the pair of cylinder decompositions shown, prove that the condition of ex. 3 holds so there are u_i, $i=1,2$ in Γ_M which are parabolic, where u_1 fixes the hor. direction and u_2 fixes direction $\theta = \frac{1}{4n}$.
Compute the corresponding matrices.

4 (cont.)

Show that there is $\mu \le 2$ and $g \in G$ s.t. $\bar{u}_1 = g u_1 g^{-1} = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$, $\bar{u}_2 = g u_2 g^{-1} = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$. Use the following diagram to conclude $\langle \bar{u}_1, \bar{u}_2 \rangle$ is a lattice. Deduce that M is a Veech surface

5. Let the notation be as in ex. 3, but (*) does not hold. Show $\overline{UM}$ is isomorphic to a torus of dimension $d = \dim_{\mathbb{Q}}(\mathrm{span}_{\mathbb{Q}}(\mu_i))$, and that the U-action on this torus is a minimal linear flow.

6. Let $U = \{u_s : s \in \mathbb{R}\}$ act on a locally compact second countable space X, and let μ be an ergodic invariant prob. measure, not supported on a single U-orbit.
(μ is a regular Borel measure).

Show that if $\mu(A)>0$ then there is $x_n \xrightarrow[n\to\infty]{} x_\infty$, where $x_n, x_\infty \in A$, and for any $1 \le i < j \le \infty$, x_i and x_j lie on different U-orbits.

7. Assume "minimal sets classification", i.e. that for any M, $\overline{UM}$ contains a surface which is a union of horizontal cylinders. Show that if $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in \Gamma_{M_1}$ then M_1 is a union of horizontal cylinders with commensurable moduli, i.e. (*) of ex. 3 is satisfied.

8. Let μ be a measure on $\mathcal{H}$. x is called <u>U-generic</u> if for all $f \in C_c(\mathcal{H})$,

$$\frac{1}{T}\int_0^T f(u_s x)\,ds \xrightarrow[T\to\infty]{} \int_{\mathcal{H}} f\,d\mu.$$

Let $a = g_{t_0}$ and suppose there is $x \in \mathcal{H}$ s.t. both x and ax are U-generic. Prove that a preserves μ (i.e. for all $f \in C_c(\mathcal{X})$

$$\int_{\mathcal{H}} f(x)\,d\mu(x) = \int_{\mathcal{H}} f(ax)\,d\mu(x)$$).