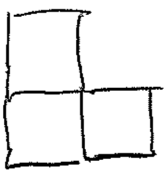
~~1372 : badge number~~
~~1083 : pin~~

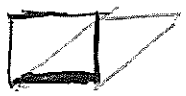
Translation surfaces and IET.

Goal: explain what happens for a family of iET which is non generic.
First explain the generic case. Basic examples and classical results.

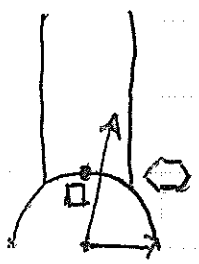
Def of ~~a IET~~ translation surface : __baby example torus__
polygons glued together $\longleftrightarrow$ holomorphic 1-form on a compact Riemann surface
Ex : octagon



cone points $\longleftrightarrow$ zeroes
cut and paste $\longleftrightarrow$ moduli space



Ex : tori (area 1) $\quad SL(2,\mathbb{R}) \diagup SL(2,\mathbb{Z})$: unit tangent bundle to the modular surface



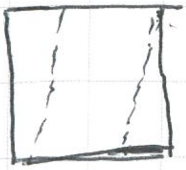
Questions : minimality, ergodicity, unique ergodicity of linear flows
tori : iff the slope is irrational

- Classical way to tackle these problems is consider the $SL(2,\mathbb{R})$ action
  g_t, h_s, r_θ (Eskin-Mirzakhani-Mohamadi).

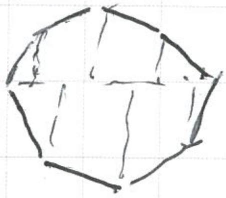
g_t is a renormalisation flow

- Rauzy induction (Veech, Zorich, Marmi-Moussa-Yoccoz)

discrete version of translation flow : IET



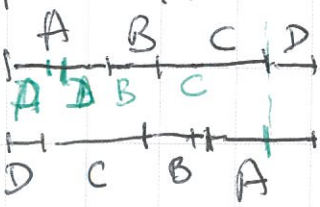
rotation



IET

$(\pi_{top}, \pi_{bot}, \vec{\lambda}) = T_{\vec{\lambda}, \pi}$ ae iet means ae λ

$\pi_{top} : \mathcal{A} \to \{1, \ldots d\}$



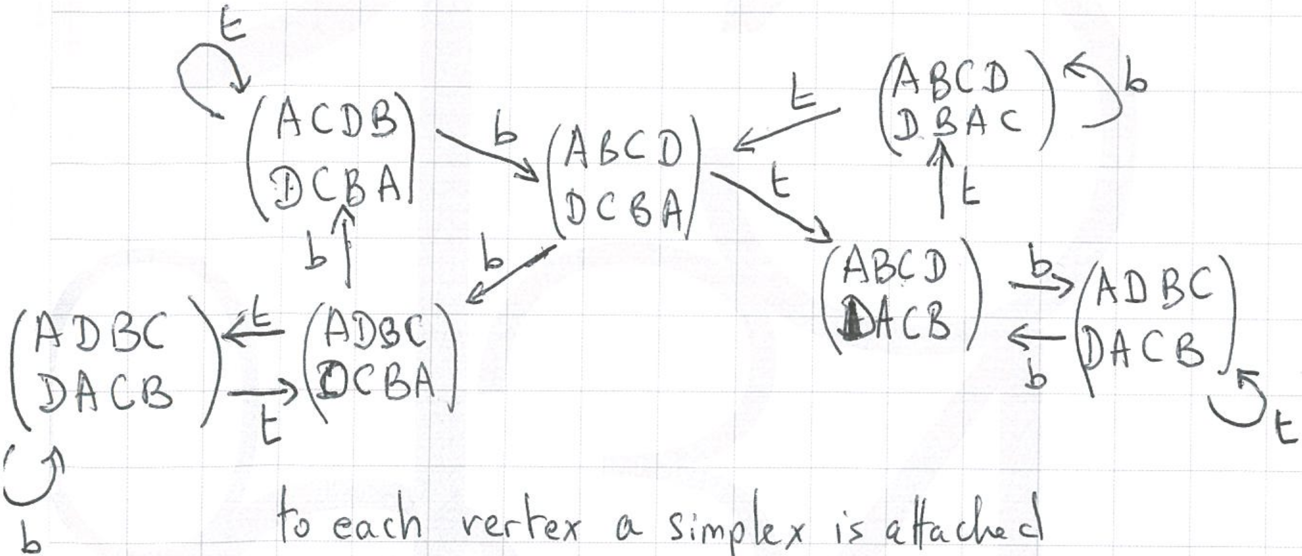
minimality : if λ_i are independent over $\mathbb{Q}$ $(\lambda_1 + \ldots \lambda_d = 1)$

UE : ae iet is UE (Masur-Veech)

$d=2$: continued fraction algorithm (acceleration...) based on renormalization same for flows

Rauzy induction (looser, winner)

$\lambda_A = \lambda'_A + \lambda'_D$ $\quad \lambda' = A^{-1}\lambda$ $\quad A \geq 0$



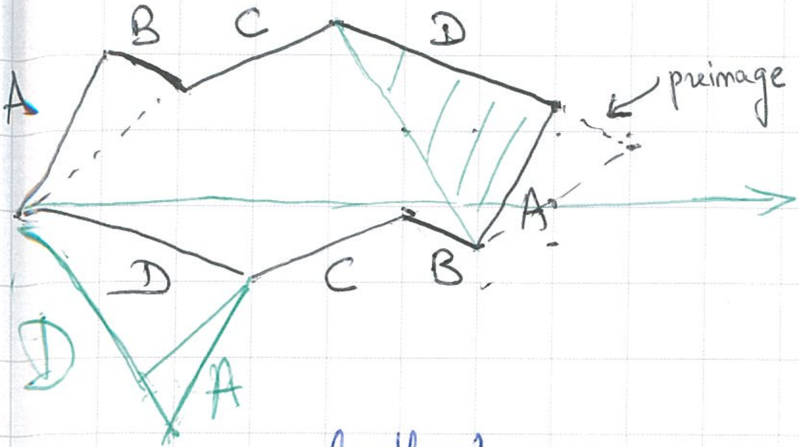
to each vertex a simplex is attached

$\Delta_\pi \xrightarrow{A_\pi} \Delta_{t(\pi)}$ onto (2 to 1 map) : projective map

symbolic dynamics (Renormalization) $A \to A$, $B \to B$, $C \to C$, $D \to AD$

Surfaces suspension data $\zeta_A = \lambda_A + i\eta_A$

$$\begin{pmatrix} A\,B\,C\,D \\ A\,C\,B\,A \end{pmatrix} \xrightarrow{t} \begin{pmatrix} A\,B\,C\,D \\ D\,A\;\,C\,B \end{pmatrix}$$



Rauzy-Veech induction is one to one

If the end point of the line is under the segment it comes from a top operation above → bottom

Canonical segment : length ≥ 1, length $R(I) < 1$

here the end point is above → bottom

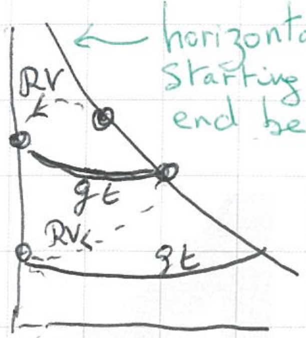
the preimage is $\begin{pmatrix} A\,C\,D\,B \\ D\,C\,A\,B \end{pmatrix}$ $\qquad \zeta_A = \zeta_A + \zeta_B$

Future $= \vec{\lambda}$

past $= \vec{\eta}$

link with Teichmüller flow

Exo: $d = 2g + r - 1$



horizontal segment of length 1 starting from a sing end below a sing

area 1 translation surface of fixed genus

first return map on a ~~...~~ codimension 1 submanifold.

Th (Masur-Veech) (Hopf argument)

Volume (transl. surf up RV induction) is finite ⇒ ergodicity of g_t

⚠ It does not mean that there is a finite absolutely continuous inv measure by the algo : ACCELERATION.

Exo: What happens for $d = 2$

__Remark__: the ergodicity of g_t is the main step to get UE for ae iet

__Special case__: self similar RV induction

loop in the R diagram

$B^{(n)} = A_n \dots A_1$ $\quad B^{(n)} > 0$

$\lambda^{(n)} = \theta^{-1}\lambda$ $\quad$ [redacted] $\theta > 1$ $\quad$ this means that $B^{-1(n)}\lambda = \theta^{-1}\lambda$

$\theta\lambda = B^{(n)}\lambda$

we choose η [redacted] st

η is the eigenvector $\leftrightarrow \theta^{-1}$

λ is the PF eigenvector of $B^{(n)}$

Rem: One has to prove that θ^{-1} is also an eigenvalue of $B^{(n)}$.
In fact $\gamma_A, \gamma_B, \gamma_C, \gamma_D$ is a basis of $H_1(X, \Sigma, \mathbb{Z})$ and
$H_1(X, \mathbb{Z}) \hookrightarrow H_1(X, \Sigma, \mathbb{Z})$.
On $H_1(X, \mathbb{Z})$ a diffeo acts as a symplectic matrix.

EXO: $f: M \to M$ compact R.S. $A: H_1(X,\mathbb{Z}) \to H_1(X,\mathbb{Z})$ respect the intersection form. For a symplectic matrix, if θ is an eigenvalue $\Rightarrow \theta^{-1}$ is also an eigenvalue.

After n steps of induction the surface is deformed by $\begin{pmatrix} \theta^{-1} & 0 \\ 0 & \theta \end{pmatrix}$. thus applying $\begin{pmatrix} e^{\tau} & 0 \\ 0 & e^{-\tau} \end{pmatrix}$ we get the initial surface
$\tau = \ln(\theta)$

~~[illegible]~~ Such homeo is called a pseudo-Anosov.
$\longleftrightarrow$ closed loop in the moduli space
Expansion in one direction / contraction in the other one.

Rem: the proof of unique ergodicity is along the same lines
we come infinitely often to a subsimplex with >0 matrix
the cone of invariant measures is contracted

$$\lambda = \dots A \dots A \dots A \dots \lambda^{(n)}$$

Rem: ae iet is rigid

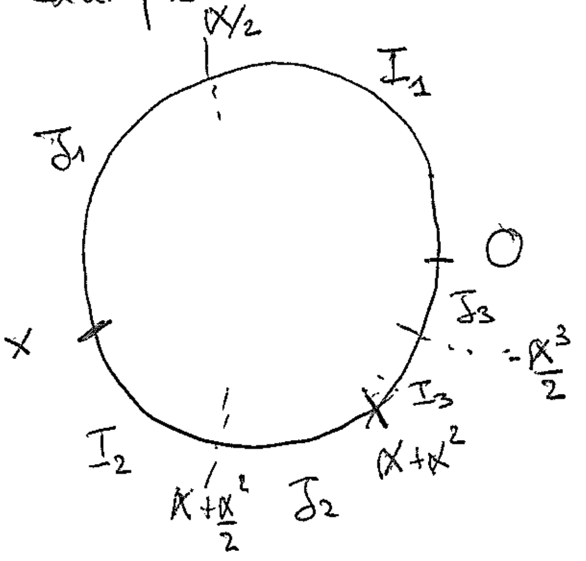
Rigid $\exists (n_k) \nearrow \infty$ st $\forall A$ $\quad \mu(T^{n_k} A \Delta A) \xrightarrow[k\to\infty]{} 0$ $\quad (T^{n_k} \to id)$

pf: $\longmapsto$ close to id + ergodicity of RV induction

Arnoux - Yoccoz iet and Arnoux - Rauzy family

Family with strange properties

The example



$1 = \alpha^3 + \alpha^2 + \alpha$

T_{AY} is the composition of 3 involutions

I_k and J_k are permuted by translation

on $\begin{cases} I_1 : x \to x + \frac{\alpha}{2} \\ J_1 : x \to x - \frac{\alpha}{2} \end{cases}$

+ rotation by π

<u>Thm</u> (Arnoux - Yoccoz) the first return map on $I_1 \cup J_1$ is T_{AY} up to scaling ie T_{AY} is self-similar.

This means, it is the horizontal part of a pseudo - Anosov map on a genus 3 surface

the dilatation is $\frac{1}{\alpha} = \beta$ $\quad \beta^3 = \beta^2 + \beta + 1$ (of odd degree)

It is minimal and uniquely ergodic

Other strange property: SAF invariant $= 0$

$SAF(T) = \sum \lambda_i \wedge_{\mathbb{Q}} t_i \in \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$

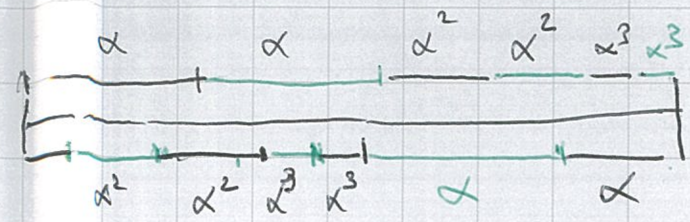
For a rotation : $SAF(R_\alpha) = 1 \wedge_{\mathbb{Q}} \alpha = 0 \Leftrightarrow \alpha \in \mathbb{Q}$

In genus 2, T ergodic $\Rightarrow SAF(T) \neq 0$

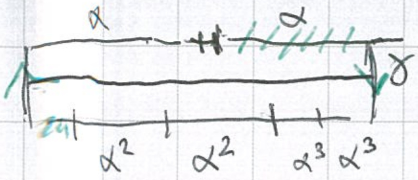
SAF is a morphism $\mathcal{G}_{iET} \to \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$

Thus $SAF(T_{AY}) = 0$

Strenner: suspension of T_{AY}



This is double cover of a non orientable surface



Moebius band with meridian γ

α identified to α by translation

...

Non orientable foliation on a non orientable surface of genus 4

Thm (Strenner) if $T \longleftrightarrow$ pseudo-Anosov map which is the lift from a pseudo-Anosov map on a non orientable surface

$SAF(T) = 0$

Symbolic coding

$I_1 \cup J_1 \to 1$

$I_2 \cup J_2 \to 2$

$I_3 \cup J_3 \to 3$

$$\begin{array}{ccc} I & \xrightarrow{T_{AY}} & I \\ c\downarrow & & \downarrow \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array}$$

C: coding

C is "onto" except for a countable set

$\Sigma = \overline{O}(u)$

Thm (Cassaigne)

T is one to one on a set of full measure

$$S: \begin{cases} 1 \to 12 \\ 2 \to 13 \\ 1 \to 1 \end{cases}$$

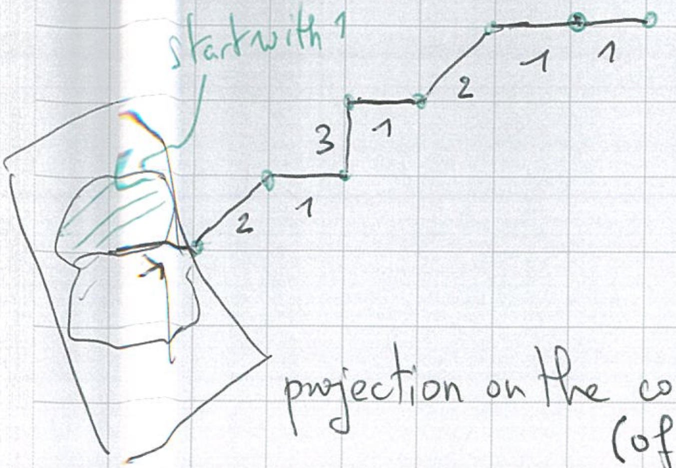
u fixed point of S

1 2 1 3 1 2

12 13 12 11213 ...

$$M_S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Broken line : abelianisation of u



bounded distance from a line since β is a Pisot number

projection on the contracting plane + closure (of the points) (or on the plane $P: x+y+z=0$)

domain exchange starts with 1 $\to$ ends with 1
if $x \in \mathcal{F}_i$ $x \to x + \pi(e_i)$

Rauzy (1982)
Arnoux (1989)

$$\begin{tikzcd}
I \arrow[r] \arrow[d, "c"'] \arrow[dd, bend right=60, "h"'] & I \arrow[d] \\
\Sigma \arrow[r, "S"] \arrow[d, "ab"'] & \Sigma \arrow[d] \\
\tilde{\mathcal{F}} \arrow[r, "E"] \arrow[d, "p"'] & \tilde{\mathcal{F}} \arrow[d] \\
\mathbb{T}^2 \arrow[r] & \mathbb{T}^2 \arrow[uu, bend right=40, "\text{Rauzy measurable isomorphism}"']
\end{tikzcd}$$

h is continuous and onto

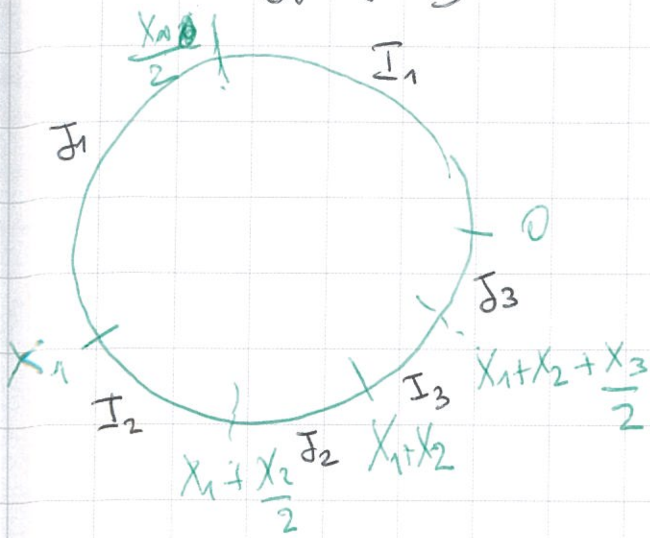
$\mathbb{T}^2 : P/\underbrace{\langle e_1 - e_2, e_1 - e_3 \rangle}_{\text{lattice in } P}$

rotation of $\mathbb{T}^2$ $\left(x \to x + \pi(e_i) \right.$ $\pi(e_i)$ is independent of

$p \circ h(I)$ is a Peano curve in $\mathbb{T}^2$ T_{Ay} is measurably conj to a rotation

Famous result: Th (Avila-Forni 2004)
ae iet is weakly mixing (no factor rotation)
Except if $g = 1$

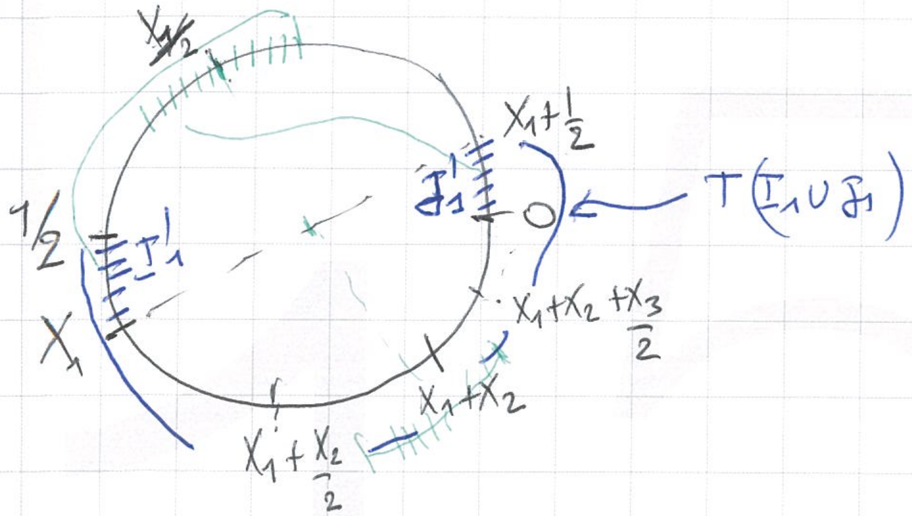
Arnoux-Rauzy family



Same construction

$x_1 > x_2 + x_3$ (or permutation)

Lemma : first return map on $T(I_1 \cup J_1)$ is a iet of the same type if we identify the extremities for the parameters $(x_1 - x_2 - x_3, x_2, x_3)$



$|I_1'| = \frac{1}{2}(\cancel{x_1} - \cancel{x_2} - \cancel{x_3}) = |J_1'|$

$I_1' \cup J_1' \subset I_1 \cup J_1 \cap T(I_1 \cup J_1) \rightarrow$ return time 1

return time 2 on $I_2 \cup J_2 \cup I_3 \cup J_3$

check the image of $I_1' \cup J_1'$

Modification of the symbolic dynamics
$1 \to 1$
$2 \to 21$
$3 \to 31$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$