THREE LECTURES ON SQUARE-TILED SURFACES

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Abstract. This text corresponds to a minicourse delivered on June 11, 12 & 13, 2018 during the summer school "Teichmüller dynamics, mapping class groups and applications" at Institut Fourier, Grenoble, France.

In this article, we cover the same topics from our minicourse, namely, origamis, Veech groups, affine homeomorphisms, and the Kontsevich–Zorich cocycle.

Contents

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1. Basic properties of origamis

This section corresponds to the content of the video available here. The reader is invited to consult Zmiaikou's Ph.D. thesis [22] for more details about the topics covered in this section.

1.1. Definitions and examples.

Let us start by defining *square-tiled surfaces*, i.e., *origamis*.

Definition 1. An *origami* is an orientable connected surface obtained from a finite collection of unit squares of $\mathbb{R}^2$ after identifications of pairs of parallel sides via adequate *translations*.

Example 2. *The square torus* $\mathbb{T}^2 = \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ *is obtained from the unit square* $[0,1] \times [0,1]$ *from identification by translations of parallel sides.*

Similarly, the L-shaped origami in Figure 1 is obtained from a collection of three unit squares by identification by translations of the sides with the same labels.

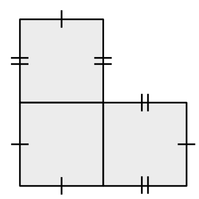

Figure 1. L-shaped origami

Remark 3. In Definition 1, by "identifications of pairs of parallel sides", we actually mean that a right vertical side of a square can only[1] be glued to a left vertical side of a square, and similarly for top and bottom sides of squares.

In particular, we *forbid* the identification of a pair of right sides of squares (for example).

Definition 4. An *origami* is a pair (X, ω), where X is a Riemann surface (complex curve) obtained as a finite cover $\pi : X \to \mathbb{T}^2 := \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$ branched only at the origin $0 \in \mathbb{T}^2$, and $\omega := \pi^*(dz)$.

These definitions of origamis are *equivalent*:

- Def. 1 $\implies$ Def. 4 because a translation is holomorphic and dz is translation-invariant;
- Def. 4 $\implies$ Def. 1 because (X, ω) is obtained by gluing by translations the squares given by the connected components of $\pi^{-1}((0,1) \times (0,1))$.

Remark 5. An origami is a particular case of the notion of *translation surfaces*: in a nutshell, a translation surface is the object obtained from a finite collection of *polygons* by gluing parallel sides by translations.

Equivalently, a translation surface is (X, ω) where X is a Riemann surface and ω is a non-trivial Abelian differential (holomorphic 1-form). Here, it is worth to recall that the nomenclature

[1] In this way, we get a full Euclidean disk around any given point.

"translation surface" comes from the fact that (X,ω) comes with an atlas of charts $X \ni z \mapsto \int_p^z \omega \in \mathbb{C}$ centered at $p \in X$ with $\omega(p) \neq 0$ such that the changes of coordinates are given by translations (because $\int_p^z \omega = \int_p^q \omega + \int_q^z \omega$). In the literature, these charts are aptly called *translation charts*.

Definition 6. An *origami* is a pair of permutations $(h, v) \in \mathrm{Sym}_N \times \mathrm{Sym}_N$ acting transitively on $\{1, \dots, N\}$.

Note that the definitions 1 and 6 are equivalent: we can label squares from 1 to N, and declare that $h(i)$, resp. $v(i)$, is the number of the neighbor to the right, resp. on the top, of the square i.

Here, the fact that h and v act transitively on $\{1, \dots, N\}$ is equivalent to the connectedness of the corresponding origami.

Remark 7. These alternative definitions of origamis indicate that origamis are rich mathematical objects which can be studied from multiple points of view (flat geometry, algebraic geometry, combinatorial group theory, etc.).

Example 8 (Regular origamis)**.** *Let G be a finite group generated by two elements r and t. The* regular origami *associated to (G, r, t) consists of taking unit squares $Sq(g)$ for each $g \in G$ and declaring*[2] *that $Sq(g \cdot r)$, resp. $Sq(g \cdot t)$, is the neighbor to the right, resp. on the top, of $Sq(g)$.*

This construction provides a rich source of origamis because many classes of finite groups generated by two elements are known, e.g.:

- *the quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$ is generated by $r = i$ and $t = j$, and the associated regular origami is the so-called* Eierlegende Wollmilchsau*;*
- *the symmetric group $G = Sym_n$ is generated by $r = (1,2)$ and $t = (1, 2, \dots, n)$;*
- *the finite group of Lie type $G = SL(2, \mathbb{F}_p)$ is generated by $r = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $t = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$*

Remark 9. Since we are interested in origamis themselves rather than particular ways of numbering their squares, our pairs of permutations (h, v) will be usually thought up to *simultaneous conjugations*, i.e., (h, v) and $(\phi h \phi^{-1}, \phi v \phi^{-1})$ determine the same origami.

1.2. **Conical singularities.** In general, the total angle around a corner of a square of an origami $\mathcal{O}$ is a *non-trivial* multiple of 2π. Any such point is called a *conical singularity* of $\mathcal{O}$.

Example 10. *The corners of all squares of the L-shaped origami in Figure 1 are identified into a conical singularity with total angle 6π.*

Example 11. *The square-tiled surface in Figure 2 has genus two and a conical singularity of total angle 6π.*

Remark 12. Conical singularities are a manifestation of the fact that a compact surface of genus $g > 1$ can not carry a flat smooth metric (by Gauss–Bonnet theorem).

[2]Our choice of multiplying by r and t on the right is a matter of convention. As we will see later, this choice has the slight advantage that an automorphism of a regular origami acts by left multiplication.

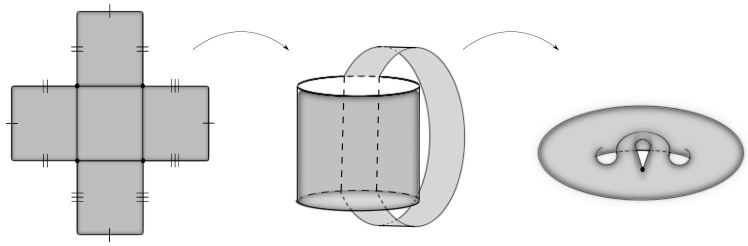

FIGURE 2. Origami with Swiss-cross shape (and genus 2).

From the combinatorial point of view, we turn around the leftmost bottom corner of a square by 2π using the *commutator* $[h, v] = vhv^{-1}h^{-1}$: see Figure 3

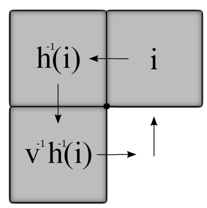

FIGURE 3. Turning by 2π around a corner.

In other terms, the conical singularities correspond to non-trivial cycles c of $[h, v]$ and the corresponding total angles are $2\pi\cdot$ length of c.

Example 13. *The L-shaped origami L in Figure 1 is associated to the permutations $h = (1, 2)(3)$ and $v = (1, 3)(2)$: see Figure 4*

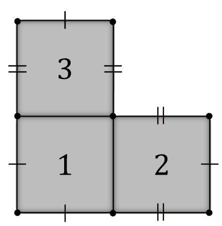

FIGURE 4. Labels for the squares of a L-shaped origami.

Since the commutator $[h, v]$ is $[h, v] = vhv^{-1}h^{-1} = (1, 3, 2)$, we get that L has an unique conical singularity of total angle $2\pi \times 3 = 6\pi$.

1.3. **Genus.** The Euler–Poincaré formula allows to express the genus g of an origami in terms of the total angles $2\pi(k_n + 1)$ around conical singularities:

$$2g - 2 = \sum k_n$$

Exercise 14. *Show this relation using triangulations for origamis.*

Example 15. *The origamis from Figures 1 and 2 both have an unique conical singularity with total angle $6\pi = 2\pi(2+1)$, hence their genera are given by the formula $2g-2=2$, i.e., $g=2$. (Of course, we already knew this fact for the origami in Figure 2 [thanks to the pictures].)*

Remark 16. A total angle of $2\pi(k+1)$ around a conical singularity means that the natural local coordinate is z^{k+1}, i.e., the associated Abelian differential is a multiple of $d(z^{k+1}) = (k+1)z^k dz$ near such a conical singularity.

1.4. Stratum and moduli spaces.

Definition 17. We say that an origami $\mathcal{O}$ belongs to the *stratum* $\mathcal{H}(k_1,\dots,k_\sigma)$ whenever the total angles of its conical singularities are $2\pi(k_n+1)$, $n=1,\dots,\sigma$.

Example 18. *The L-shaped origami in Figure 1 belongs to $\mathcal{H}(2)$.*

Proposition 19. *An origami in $\mathcal{H}(k_1,\dots,k_\sigma)$ is tiled by at least $\sum_{n=1}^{\sigma}(k_n+1)$ squares.*

Proof. We saw that an origami in $\mathcal{H}(k_1,\dots,k_\sigma)$ is given by a pair of permutations $(h,v) \in \mathrm{Sym}_N \times \mathrm{Sym}_N$ whose commutator $\mathrm{Sym}_N \ni [h,v]$ has σ non-trivial cycles of lengths k_n+1, $n=1,\dots,\sigma$. Therefore,

$$N \geq \sum_{n=1}^{\sigma}(k_n+1)$$

□

Remark 20. This proposition implies that an origami in $\mathcal{H}(2)$ is made out of 3 squares at least. Thus, in a certain sense, the origami in Figure 1 is one of the smallest possible origamis in $\mathcal{H}(2)$.

The nomenclature "stratum" comes from the fact that the *moduli space of translation surfaces of genus g* is naturally stratified by fixing the total angles around conical singularities.

The basic idea behind the construction of moduli spaces of translation surfaces is simple: we want to declare that two translation surfaces deduced from each other by cutting and *gluing by translations* are the same.

Example 21. *By cutting and pasting by translations as in Figure 5 we see that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbb{T}^2 = \mathbb{T}^2$ at the level of moduli space.*

The discussion of moduli spaces is out of the scope of these notes: the reader can consult [9] for more explanations. For this reason, we close this subsection with the following remarks about moduli spaces of translation surfaces:

- The strata $\mathcal{H}(k_1,\dots,k_\sigma)$ are complex orbifolds and their local (period) coordinates are related to the complex numbers (vectors in $\mathbb{R}^2$) representing sides of polygons;

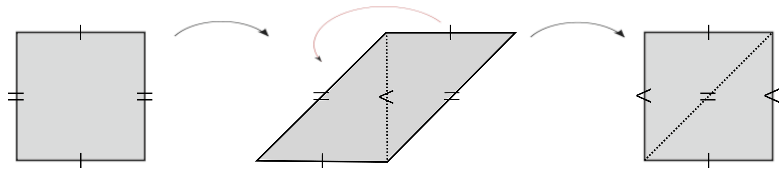

FIGURE 5. Cutting and pasting after shear.

- Square-tiled surfaces correspond to *integral* points of strata in a certain sense: cf. Gutkin–Judge paper [10];
- Similarly to the fact (going back to Gauss) that the area of large balls and, consequently, the area of the unit ball is related to counting integral points, the volumes of the strata $\mathcal{H}(k_1, \dots, k_\sigma)$ of moduli spaces of translation surfaces (with respect to the so-called *Masur–Veech measures*) are related to counting square-tiled surfaces.

  Moreover, these counting problems leads to beautiful topics such as multi-zeta values, quasi-modular forms, etc.: cf. Zorich [23], Eskin–Okounkov [6], etc.
- Strata are *not* always connected, but their connected components were completely classified by Kontsevich–Zorich [13] in 2003; in particular, $\mathcal{H}(k_1, \dots, k_\sigma)$ has 3 connected components at most.

1.5. **Reduced and primitive origamis.** The *period lattice* $\mathrm{Per}(\omega)$ of an origami (M, ω) is the lattice spanned by the *holonomy vectors* $\int_\gamma \omega$ of paths γ whose endpoints are conical singularities of (M, ω).

Definition 22. An origami (M, ω) is *reduced* whenever its period lattice $\mathrm{Per}(\omega)$ is $\mathbb{Z} \oplus i\mathbb{Z}$.

Equivalently, an origami $\pi : \mathcal{O} \to \mathbb{T}^2$ is reduced whenever any factorization $\pi = p \circ \pi'$ with $\pi' : \mathcal{O} \to \mathbb{T}^2$ and $p : \mathbb{T}^2 \to \mathbb{T}^2$ is trivial, i.e., p has degree 1.

Intuitively, a reduced origami does not have "unnecessary" squares: for instance, if we replace the unit squares tiling the origami $\mathcal{O}_1$ in Figure 1 by squares with sides of length 2, then we get a L-shaped origami $\mathcal{O}_2$ tiled by 12 unit squares[3] which is not reduced; of course, we see that $\mathcal{O}_2$ is not different from $\mathcal{O}_1$ except for the fact that there are "too many" unit squares in its construction.

Standing assumption. From now on, all origamis are assumed to be *reduced* unless *explicitely* stated otherwise.

Remark 23. We can "reduce" an arbitrary origami via *scaling*.

Definition 24. An origami is *primitive* if it is *not* a non-trivial cover of another origami.

[3]Since each 2×2 square is divided into four unit squares.

A primitive origami is reduced, but the converse is not true in general: the square-tiled surface in Figure 6 is reduced, but it is not primitive because it is a double cover of the L-shaped origami in Figure 1.

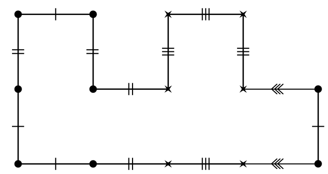

FIGURE 6. Reduced and non-primitive origami.

Combinatorially speaking, the primitivity of an origami corresponds to the primitivity (in the sense of combinatorial group theory) of the associated permutation subgroup.

More precisely, let $\mathcal{O}$ be an origami defined by two permutations $(h, v) \in \mathrm{Sym}_N \times Sym_N$, consider the set $Sq(\mathcal{O}) \simeq \{1, \dots, N\}$ of the squares tiling $\mathcal{O}$, and denote by $G = \sigma(\mathcal{O})$ the associated permutation of $\mathrm{Sym}(Sq(\mathcal{O}))$: in a nutshell, G is the subgroup of $\mathrm{Sym}(Sq(\mathcal{O})) \simeq \mathrm{Sym}_N$ generated by the permutations h and v.

In this setting, it is possible to show that $\mathcal{O}$ is primitive if and only if $G = \sigma(\mathcal{O})$ is primitive in the sense that there is no *block* $\Delta \subset Sq(\mathcal{O})$, i.e., a subset of cardinality $1 < \#\Delta < \#\mathrm{Sq}(\mathcal{O})$ with $\alpha(\Delta) = \Delta$ or $\alpha(\Delta) \cap \Delta = \emptyset$ for each $\alpha \in G$.

Theorem 25 (Zmiaikou)**.** *A primitive origami $\mathcal{O} \in \mathcal{H}(k_1, \dots, k_\sigma)$ tiled by*

$$N \geq \left(2 \sum_{n=1}^{\sigma} (k_n + 1)\right)^2$$

squares has associated permutation group $\sigma(\mathcal{O}) = Alt(Sq(\mathcal{O}))$ or $Sym(Sq(\mathcal{O}))$.

Proof. The features of primitive subgroups of permutations groups is a classical topic in combinatorial group theory. In particular:

- Jordan showed in 1873 that a primitive subgroup G of Sym_m containing a cycle of prime order $p \leq n - 3$ equals to Alt_m or Sym_m;
- more recently, some results obtained by Babai (in 1982) and Pyber (in 1991) imply that a primitive subgroup G of Sym_m not containing the alternating group Alt_m satisfies

$$m < 4 \left(\min_{\alpha \in G \setminus \{id\}} \#\mathrm{supp}(\alpha) \right)^2$$

In our context of primitive origamis $\mathcal{O}$, the desired theorem follows directly from the results of Babai and Pyber because $\sigma(\mathcal{O})$ contains the commutator $[h, v]$ of a pair of permutations determining $\mathcal{O}$ and the support of $[h, v]$ has cardinality $\geq \sum_{n=1}^{\sigma} (k_n + 1)$ whenever $\mathcal{O} \in \mathcal{H}(k_1, \dots, k_\sigma)$. □

Remark 26.

- $\sigma(\mathcal{O})$ is often used to distinguish origamis;

- the previous theorem says that, in each fixed stratum $\mathcal{H}$, for all but finitely many primitive origamis $\mathcal{O} \in \mathcal{H}$, the subgroup $\sigma(\mathcal{O})$ takes only two types of values (namely, Alt or Sym).

2. $SL(2,\mathbb{R})$-ORBITS AND HOMOLOGY OF ORIGAMIS

This section corresponds to the content of the video available here. The reader is invited to consult Hubert–Schmidt survey [12] and the article [14] for more explanations and/or references about the topics covered in this section.

2.1. **Action of $SL(2,\mathbb{R})$ on origamis.** The linear action of $g \in SL(2,\mathbb{R}^2)$ on $\mathbb{R}^2$ induces a natural action on origamis, namely, we apply g to each of the squares tiling a given origami and we keep identifying by translations the same[4] pairs of sides. In particular, $g(\mathcal{O})$ is a *translation surface* of area N whenever $\mathcal{O}$ is an origami tiled by N unit squares in $\mathbb{R}^2$.

Example 27. *The action of the matrix* $h = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$ *on the L-shaped in Figure 1 is shown in Figure 13.*

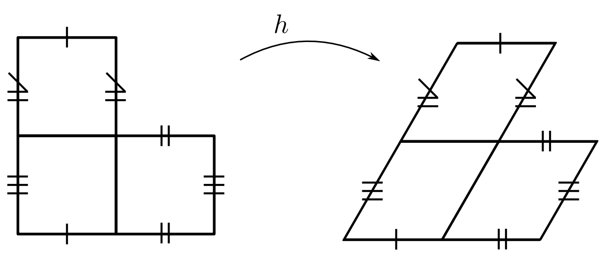

FIGURE 7. Example of $SL(2,\mathbb{R})$-action.

The translation surface $g(\mathcal{O})$ is *not* an origami in general. Nevertheless, $g(\mathcal{O})$ is an origami *whenever* $g \in SL(2,\mathbb{Z})$: indeed, the cover $\mathcal{O} \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ associated to the origami $\mathcal{O}$ induces a cover $g(\mathcal{O}) \to g(\mathbb{R}^2)/g(\mathbb{Z}^2) = \mathbb{R}^2/g(\mathbb{Z}^2)$ and $SL(2,\mathbb{Z})$ is the stabilizer of $\mathbb{Z}^2$ for the linear action of $SL(2,\mathbb{R})$ in $\mathbb{R}^2$.

Example 28. *The cutting and pasting process in Figure 5 shows that* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(\mathbb{T}^2)$.

A similar argument also shows that $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in SL(\mathbb{T}^2)$.

Remark 29. It is possible to show that $SL(2,\mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$: see Serre's book [20].

[4]This is well-defined because the linear action of g respects the notion of parallelism.

2.2. **Veech groups.** The *Veech group* $SL(\mathcal{O})$ of an origami $\mathcal{O}$ is the stabilizer of $\mathcal{O}$ in $SL(2,\mathbb{R})$.

Note that $SL(\mathcal{O})$ is a subgroup of $SL(2,\mathbb{Z})$ when $\mathcal{O}$ is reduced. In fact, an origami $(\mathcal{O},\omega)$ is a cover $\mathcal{O}\to\mathbb{C}/\mathrm{Per}(\omega)$ given by

$$\mathcal{O}\ni z\mapsto\int_p^z\omega\in\mathbb{C}/\mathrm{Per}(\omega)$$

where p is a conical singularity and $\mathrm{Per}(\omega)$ is the period lattice. On the other hand, if $\mathcal{O}$ is reduced, then $\mathrm{Per}(\omega)=\mathbb{Z}\oplus i\mathbb{Z}$. Therefore, $g\in SL(\mathcal{O})$ stabilizes $\mathrm{Per}(\omega)=\mathbb{Z}\oplus i\mathbb{Z}$, so that $g\in SL(2,\mathbb{Z})$.

Example 30. *The Veech group of* $\mathbb{T}^2$ *is* $SL(2,\mathbb{Z})$ *(compare with Example 28 and Remark 29).*

2.3. $SL(2,\mathbb{Z})$**-orbits.** In practice, we exploit the fact that $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $S=\begin{pmatrix}1&0\\1&1\end{pmatrix}$ generate $SL(2,\mathbb{Z})$ to study Veech groups and $SL(2,\mathbb{Z})$-orbits of origamis.

Example 31. *The* $SL(2,\mathbb{Z})$*-orbit of the L-shaped origami from Figure 1 is displayed in Figure 8.*

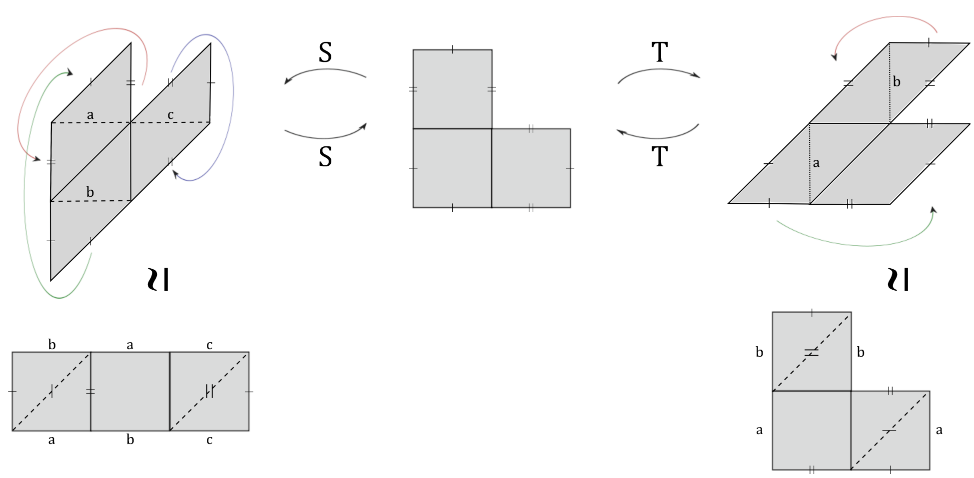

FIGURE 8. $SL(2,\mathbb{Z})$-orbit of a L-shaped origami.

The problem of computing $SL(2,\mathbb{Z})$-orbits of origamis can be solved in a purely *combinatorial* way[5] More precisely, let $\mathcal{O}$ be an origami determined by two permutations $h,v\in\mathrm{Sym}_N$. Note that $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ acts on a square labelled i as in Figure 9.

Hence, on $T(\mathcal{O})$, the neighbor to the right of n is $h(n)$ and the neighbor on the top of n is $vh^{-1}(n)$. In other words, the action of T on pairs of permutations consists in sending $\mathcal{O}=(h,v)$ to $T(\mathcal{O})=(h,vh^{-1})$.

[5]In particular, it is not surprising that $SL(2,\mathbb{Z})$-orbits can be determined with computer programs: see Vincent Delecroix's webpage for more details.

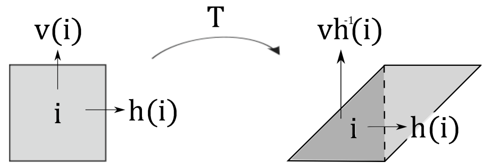

FIGURE 9. Action of T on pairs of permutations.

Similarly, the action of $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ on pairs of permutations can be deduced by symmetry (i.e., exchanging the roles of h and v): it turns out that S maps $\mathcal{O} = (h, v)$ to $S(\mathcal{O}) = (hv^{-1}, v)$.

Finally, these calculations are always performed while keeping in mind that pairs of permutations (h, v) are taken *modulo* simultaneous conjugations (cf. Remark 9).

Remark 32. T and S are particular instances of *Nielsen transformations*.[6] Note also that our formulas for the actions of T and S on pairs of permutations imply that the permutation subgroup $\sigma(\mathcal{O})$ associated to an origami $\mathcal{O}$ is $SL(2,\mathbb{Z})$-invariant.

Proposition 33. *Veech groups of (reduced) origamis are finite-index subgroups of $SL(2,\mathbb{Z})$.*

Proof. Our description of the actions of the generators $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ of $SL(2,\mathbb{Z})$ on pairs of permutations says that $SL(2,\mathbb{Z})$-orbits of origamis are finite (because $\#(\mathrm{Sym}_N \times \mathrm{Sym}_N) = (N!)^2$), so that Veech groups are finite-index subgroups of $SL(2,\mathbb{Z})$ (because they are stabilizers[7] of origamis). □

Example 34. *The L-shaped origami $\mathcal{O}_0$ from Figure 1 can be described by the permutations $h_0 = (1,2)(3)$ and $v_0 = (1,3)(2)$.*

Hence:

- *the origami $\mathcal{O}_1 = T(\mathcal{O}_0)$ is determined by the permutations $h_1 = h_0$ and $v_1 = v_0 h_0^{-1} = (1,2,3)$*
- *the origami $S(\mathcal{O}_1)$ is given by the permutations $h = h_1 v_1^{-1} = (1,3)(2)$ and $v = v_1$; as it turns out, h and v are simultaneously conjugated to h_1 and v_1: indeed, $h_1 = \phi h \phi^{-1}$ and $v_1 = \phi v \phi^{-1}$ where $\phi(1) = 2$, $\phi(2) = 3$ and $\phi(3) = 1$; thus, $S(\mathcal{O}_1) = \mathcal{O}_1$.*

Exercise 35. *Compute the $SL(2,\mathbb{Z})$-orbit of the Eierlegende Wollmilchsau (cf. Example 8 and Figure 10).*

Among the important open problems about Veech groups and $SL(2,\mathbb{Z})$-orbits of origamis, we can mention:

[6]Useful operations invented by Nielsen to recognize generating subsets of free groups.

[7]I.e., the quotient of $SL(2,\mathbb{Z})$ by a Veech group is a $SL(2,\mathbb{Z})$-orbit.

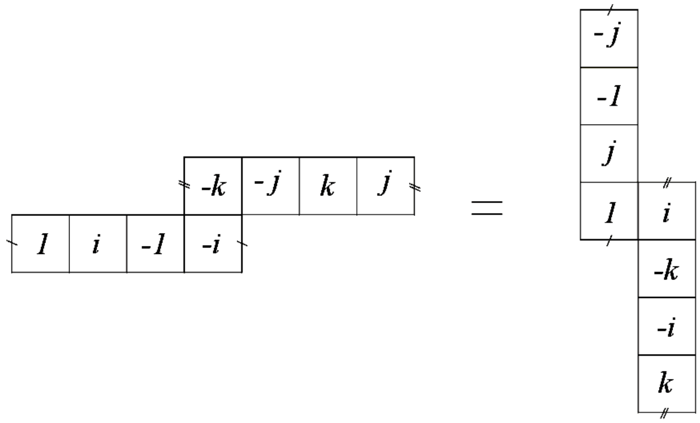

FIGURE 10. Eierlegende Wollmilchsau.

- classification of $SL(2,\mathbb{Z})$-orbits: Hubert–Lelièvre [11] and McMullen [17] showed that the associated permutation subgroup $\sigma(\mathcal{O})$ is a *complete* invariant for $\mathcal{O}\in\mathcal{H}(2)$, but we have only some conjectures (supported by numerical evidence) about $SL(2,\mathbb{Z})$-orbits of origamis in $\mathcal{H}(1,1)$, $\mathcal{H}(4)$, etc.
- classification of Veech groups: Schmithüsen showed that principal congruence subgroups of $SL(2,\mathbb{Z})$ are Veech groups of origamis (see [18]), but the Veech groups of origamis in $\mathcal{H}(2)$ virtually never contain principal congruence subgroups (see [19]); in general, Ellenberg–McReynolds [3] proved that any finite-index subgroup of the principal congruence group of level 2 containing $-\mathrm{Id}$ is the Veech group of an origami, but we do not know examples of finite-index subgroups of $SL(2,\mathbb{Z})$ which are not Veech groups of origamis.

2.4. **Automorphisms.** An *automorphism* of an origami $\mathcal{O}=(X,\omega)$ is a biholomorphism $A: X\to X$ respecting ω.

Remark 36. Since $\omega = dz$ locally outside zeros, an automorphism acts by *translations* on squares.

The group of automorphisms of an origami $\mathcal{O}$ is denoted by $\mathrm{Aut}(\mathcal{O})$. By Remark 36, $\mathrm{Aut}(\mathcal{O})$ is always a *finite* group.

On one hand, regular origamis possess rich groups of automorphisms:

Example 37. *Given G a finite group generated by two elements r and t, denote by $\mathcal{O}_{G,r,t}$ the corresponding regular origami constructed in Example 8.*

For $a\in G$, consider the map φ_a obtained by translating the square $Sq(g)$ labelled $g\in G$ to the square $Sq(a\cdot g)$. Since "associativity is commutativity"[8], we see that φ_a (= translation by multiplication by a on the left of g) respects the identifications (=multiplications by r and t on the right of g), so that φ_a is an automorphism of $\mathcal{O}_{G,r,t}$.

[8]Denis Sullivan uses this phrase to point out that "associativity" simply means that the operations of taking left and right multiplications commute.

In particular, G is naturally isomorphic to a subgroup of $Aut(\mathcal{O}_{G,r,t})$.

Exercise 38. *Check that $G \simeq Aut(\mathcal{O}_{G,r,t})$ in Example 37.*

On the other hand, the automorphisms groups of origamis in minimal strata are trivial:

Proposition 39. *$Aut(\mathcal{O}) = \{Id\}$ whenever $\mathcal{O} \in \mathcal{H}(2g-2)$.*

Proof. Otherwise, we can take $A \in \mathrm{Aut}(\mathcal{O}) \setminus \{\mathrm{Id}\}$ and consider the non-trivial finite cyclic cover

$$\mathcal{O} \to \mathcal{O}' = \mathcal{O}/\langle A \rangle$$

Denote by $* \in \mathcal{O}$ and $*' \in \mathcal{O}'$ the unique conical singularities of these origamis. A small loop around $*'$ is a product of commutators in $\pi_1(\mathcal{O}')$, so that its lift to $\mathcal{O}$ via the Abelian cover $\mathcal{O} \to \mathcal{O}'$ is also a small loop. In other terms, the non-trivial finite cover $\mathcal{O} \to \mathcal{O}'$ would be *unramified*, and, *a fortiori*, $\mathcal{O} \notin \mathcal{H}(2g-2)$, a contradiction. □

2.5. **Affine homeomorphisms.** An *affine homeomorphism* $f : (X, \omega) \to (X, \omega)$ of an origami (X, ω) is an orientation-preserving homeomorphism respecting the conical singularities which is *affine* in the translation charts $z \mapsto \int_p^z \omega$ from Remark 5.

The group of affine homeomorphisms of (X, ω) is denoted by $\mathrm{Aff}(X, \omega)$.

An affine map on $\mathbb{R}^2$ is the action of a linear map followed by a translation. This description can be used to show that

$$\{\mathrm{Id}\} \to \mathrm{Aut}(\mathcal{O}) \to \mathrm{Aff}(\mathcal{O}) \to SL(\mathcal{O}) \to \{\mathrm{Id}\}$$

is a *short exact sequence* (where $\mathrm{Aut}(\mathcal{O}) \to \mathrm{Aff}(\mathcal{O})$ is the inclusion and $\mathrm{Aff}(\mathcal{O}) \to SL(\mathcal{O})$ is the "derivative" map sending an affine homeomorphism to its linear part).

In particular, $\mathrm{Aff}(\mathcal{O})$ is isomorphic to $SL(\mathcal{O})$ for an origami with $\mathrm{Aut}(\mathcal{O}) = \{\mathrm{Id}\}$.

Remark 40. Given an origami $\mathcal{O}$, its group of affine homeomorphisms $\mathrm{Aff}(\mathcal{O})$ is the stabilizer in the mapping class group of its $SL(2, \mathbb{R})$-orbit inside Teichmüller space.

It was shown by Smillie that the $SL(2, \mathbb{R})$-orbit inside moduli space of an origami $\mathcal{O}$ is a *closed* suborbifold isomorphic to $SL(2, \mathbb{R})/SL(\mathcal{O})$.

The $SL(2, \mathbb{R})$-orbit of an origami $\mathcal{O}$ is called *Teichmüller curve* because $SL(2, \mathbb{R})/SL(\mathcal{O}) \simeq T^1\mathbb{H}/SL(\mathcal{O})$ is the unit cotangent bundle to the Riemann surface (complex curve) $\mathbb{H}/SL(\mathcal{O})$.

2.6. **Homology of origamis.** Let $\mathcal{O}$ be an origami associated to a pair of permutations $h, v \in \mathrm{Sym}_N$. Denote by Σ the subset of $\mathcal{O}$ consisting of all points at corners of squares of $\mathcal{O}$.

For each square $Sq(n)$, $n \in \{1, \dots, N\}$, of $\mathcal{O}$, let us denote by σ_n the oriented (from left to right) horizontal cycle corresponding to the bottom side of $Sq(n)$, and ζ_n the oriented (from bottom to top) vertical cycle corresponding to the left side of $Sq(n)$: see Figure 11. Note that, by definition, the top side of $Sq(n)$ is $\sigma_{v(n)}$ and the right side of $Sq(n)$ is $\zeta_{h(n)}$.

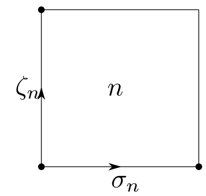

FIGURE 11. Relative homology cycles of an origami

The *relative* homology group $H_1(\mathcal{O}, \Sigma, \mathbb{R})$ is the module spanned by the cycles σ_n, ζ_n satisfying the relations[9]

$$\sigma_n + \zeta_{h(n)} = \zeta_n + \sigma_{v(n)}$$

The *boundary* of an oriented cycle σ from a point A to a point B is the difference $\partial\sigma = B - A$. The vector space $H_1(\mathcal{O}, \mathbb{R})$ consisting of all cycles in $H_1(\mathcal{O}, \Sigma, \mathbb{R})$ with zero boundary is the *absolute* homology of $\mathcal{O}$. It is known that $H_1(\mathcal{O}, \mathbb{R})$ has dimension $2g$ where g is the genus of $\mathcal{O}$.

The absolute homology $H_1(\mathcal{O}, \mathbb{R})$ has a natural structure of *symplectic* vector space with respect to the so-called *intersection form*. In a nutshell, given two simple oriented closed curves α and β avoiding conical singularities and intersecting transversely, we attribute a sign $\varepsilon_p = \pm 1$ at each point $p \in \alpha \cap \beta$ depending on the orientation of the basis $\{\dot{\alpha}(p), \dot{\beta}(p)\}$ of $T_p\mathcal{O}$, and we define the intersection of α and β by $(\alpha, \beta) = \sum\limits_{p \in \alpha\cap\beta} \varepsilon_p$. Finally, we extend $(.,.)$ to $H_1(\mathcal{O}, \mathbb{R})$ in a linear way.

The *tautological plane* $H_1^{st}(\mathcal{O})$ is the plane in absolute homology spanned by the cycles

$$\sigma := \sum_{n \in \mathrm{Sq}(\mathcal{O})} \sigma_n \quad \text{and} \quad \zeta := \sum_{n \in \mathrm{Sq}(\mathcal{O})} \zeta_n$$

Observe that $H_1^{st}(\mathcal{O})$ is a symplectic plane because $(\sigma, \zeta) = N = \#\mathrm{Sq}(\mathcal{O})$ (as σ_n and ζ_n intersects only once in the middle of the square n). Thus,

$$H_1(\mathcal{O}, \mathbb{R}) = H_1^{st}(\mathcal{O}) \oplus H_1^{(0)}(\mathcal{O})$$

where $H_1^{(0)}(\mathcal{O})$ is the symplectic orthogonal of $H_1^{st}(\mathcal{O})$.

Exercise 41. *Check that* $H_1^{(0)}(\mathcal{O})$ *is the* $(2g-2)$*-dimensional vector space of* zero holonomy *cycles, i.e.,* $H_1^{(0)}(\mathcal{O}) = \left\{\gamma \in H_1(\mathcal{O}, \mathbb{R}) : \int_\gamma \omega = 0\right\}$.

Remark 42. The decomposition $H_1(\mathcal{O}, \mathbb{R}) = H_1^{st}(\mathcal{O}) \oplus H_1^{(0)}(\mathcal{O})$ is defined over $\mathbb{Z}$.

[9]Accounting for the fact that the cycle $\sigma_n + \zeta_{h(n)} - \sigma_{v(n)} - \zeta_n$ vanishes in homology because it bounds a piece of surface (namely, the interior of $Sq(n)$).

3. Actions on homologies of origamis

This section corresponds to the content of the video available here. The reader is invited to consult Hubert–Schmidt survey [12] and the articles [15], [16] for more explanations.

An inspection of Figure 9 reveals that the action on *relative* homologies of (a certain affine homeomorphism with linear part) $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with respect to the generating cycles $\{\sigma_n, \zeta_n\}$ on $\mathcal{O}$ and $\{\sigma'_n, \zeta'_n\}$ on $T(\mathcal{O})$ is

$$T(\sigma_n) = \sigma'_n \quad \text{and} \quad T(\zeta_n) = \zeta_n + \sigma'_{vh^{-1}(n)}$$

where (h, v) is a pair of permutations representing $\mathcal{O}$.

Similarly, the action of (a certain affine homeomorphism with linear part) $S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ on the relative homologies of $\mathcal{O}$ and $S(\mathcal{O})$ is

$$S(\sigma_n) = \sigma''_n + \zeta''_{hv^{-1}(n)} \quad \text{and} \quad S(\zeta_n) = \zeta''_n$$

In principle, the action on *absolute* homologies can be deduced from these formulas. In practice, it is sometimes more convenient to exploit the geometry of the examples at hand.

3.1. Cylinder decompositions and Dehn twists.

The actions of T and S on $H_1(\mathbb{T}^2) = H_1^{st}(\mathbb{T}^2)$ with respect to the basis $\{\sigma, \zeta\}$ is very simple:

$$T(\sigma) = \sigma, \quad T(\zeta) = \zeta + \sigma, \quad S(\sigma) = \sigma + \zeta, \quad S(\zeta) = \zeta,$$

that is, the matrices representing the actions of T, resp. S, on $H_1(\mathbb{T}^2)$ with respect to the basis $\{\sigma, \zeta\}$ are $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, resp. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Actually, the same fact is true in general: the actions of T and S on the *tautological* planes of origamis are represented by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \tag{3.1}$$

On the other hand, the actions on the zero holonomy subspaces $H_1^{(0)}$ are extremely sensitive to the geometry of the origamis at hand.

For example, let us consider the origami L in Figure 12.

The absolute cycles $\{\sigma_1, \sigma_2, \zeta_1, \zeta_2\}$ indicated in Figure ??? form a basis of $H_1(L, \mathbb{R})$. In fact, we know that L has genus 2, so that $H_1(L, \mathbb{R})$ has dimension $4 = 2 \times 2$. Hence, our task is reduced to show that:

Exercise 43. *Check that $\sigma_1, \sigma_2, \zeta_1, \zeta_2$ are linearly independent.*[10]

[10] Hint: Use the intersection form, e.g., $\sigma_2 = \lambda\sigma_1$ would imply $1 = (\sigma_2, \zeta_1) = \lambda(\sigma_1, \zeta_1) = \lambda \cdot 0 = 0$, a contradiction.

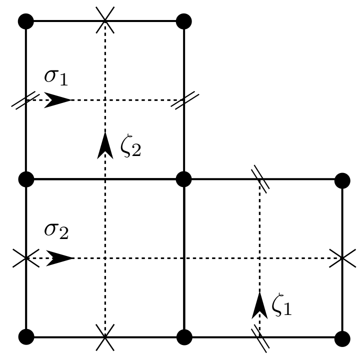

FIGURE 12. Cycles in the absolute homology of a L-shaped origami

Let us now compute the action of $T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ with respect to the basis $\{\sigma_1, \sigma_2, \zeta_1, \zeta_2\}$.

Remark 44. We chose T^2 because $T^2 \in SL(L)$ (cf. Figure 8).

For this sake, we need to recall the notions of *cylinder decompositions* and *Dehn twists*.

Definition 45. A *cylinder* is a maximal collection of parallel closed geodesics.

In Figure 13, we indicated two (white and grey) cylinders in L consisting of horizontal closed geodesics.

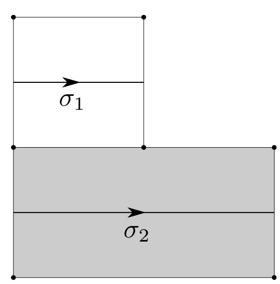

FIGURE 13. Horizontal cylinders on a L-shaped origami.

In general, any origami decomposes as the union of finitely many cylinders of closed geodesics in any fixed direction of rational slope. In the literature, this is called the *cylinder decomposition*[11] of the origami in the prescribed rational direction: for instance, Figure 13 illustrates the cylinder decomposition in the horizontal direction of a L-shaped origami.

A horizontal cylinder is isometric to a rectangle $[0, \ell] \times [0, h]$ whose vertical sides $\{0\} \times [0, h]$ and $\{\ell\} \times [0, h]$ are identified by translation.

[11]Cylinder decompositions are a particular case of the so-called *Thurston–Veech construction*.

The parabolic matrix $t = \begin{pmatrix} 1 & \ell/h \\ 0 & 1 \end{pmatrix}$ acts on the horizontal cylinder $[0,\ell]\times[0,h]$ in a simple way: it shears this cylinder C into a parallelogram P in such a way that it is possible to cut P into two triangles and paste them back by translation in order to recover C (compare with Figure 5).

In this process, the *waist curve* $w = [0,\ell]\times\{h/2\}$ of the horizontal cylinder $[0,\ell]\times[0,h]$ is preserved by the parabolic matrix t, but a vertical cycle γ connecting the bottom side $[0,\ell]\times\{0\}$ to the top side $[0,\ell]\times\{h\}$ is sent by t to

$$t(\gamma) = \gamma + w$$

This operation is an example of *Dehn twist* about w.

Coming back to the calculation of the action of T^2 on $H_1(L,\mathbb{R})$, we have that T^2 acts by Dehn twists on the white and grey horizontal cylinders in Figure 13. Hence,

$$T^2(\sigma_1) = \sigma_1, \quad T^2(\sigma_2) = \sigma_2, \quad T^2(\zeta_1) = \zeta_1 + \sigma_2, \quad T^2(\zeta_2) = \zeta_2 + \sigma_2 + 2\sigma_1.$$

Note that the cycles σ and ζ spanning the tautological plane $H_1^{st}(L)$ are

$$\sigma = \sigma_1 + \sigma_2, \quad \zeta = \zeta_1 + \zeta_2$$

It follows from the previous formulas that $T^2(\sigma) = \sigma$ and $T^2(\zeta) = \zeta + 2\sigma$. In particular, we recover the fact mentioned in (3.1) that T^2 acts on the tautological plane by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

The zero holonomy space $H_1^{(0)}(L)$ is generated by the cycles $\Sigma = \sigma_2 - 2\sigma_1$ and $Z = \zeta_2 - 2\zeta_1$. Again, it follows from the previous formulas that T^2 acts on $H_1^{(0)}(L)$ as

$$T^2(\Sigma) = \Sigma \quad \text{and} \quad T^2(Z) = Z - \Sigma.$$

In other terms, $T^2|_{H_1^{(0)}(L)}$ acts by the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

In summary, the matrix of the action of T^2 on $H_1(L,\mathbb{R})$ with respect to the basis $\{\sigma,\zeta,\Sigma,Z\}$ is

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Remark 46. This strategy of understanding complicated actions (like $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = T^2S^2$) on origamis via cylinder decompositions and Dehn twists goes back to Thurston.

3.2. **Kontsevich–Zorich cocycle.** In general, the action on homology of affine homeomorphisms of an origami $\mathcal{O}$ gives rise[12] to a *representation*

$$\rho : \mathrm{Aff}(\mathcal{O}) \to Sp(H_1(\mathcal{O},\mathbb{R}))$$

[12]Here, we are using the fact that the symplectic intersection form on homology is invariant under the natural action of an orientation-preserving homeomorphism.

Note that ρ is *reducible*: $\text{Aff}(\mathcal{O})$ respects the decomposition $H_1(\mathcal{O},\mathbb{R}) = H_1^{st}(\mathcal{O}) \oplus H_1^{(0)}(\mathcal{O})$.

The image of $\rho|_{H_1^{(0)}(\mathcal{O})}$ is the *Kontsevich–Zorich monodromy* of $\mathcal{O}$. The nomenclature comes from the following well-known construction (cf. Figure 14): given the representation ρ, we build a *flat bundle* with a natural $SL(2,\mathbb{R})$-action by considering the trivial bundle $SL(2,\mathbb{R}) \times H_1(\mathcal{O},\mathbb{R})$ where $g \in SL(2,\mathbb{R})$ acts by $g(h,\gamma) = (gh,\gamma)$, and by taking the quotient of $SL(2,\mathbb{R}) \times H_1(\mathcal{O},\mathbb{R})$ by the *diagonal action* of $\text{Aff}(\mathcal{O})$ (where $f \in \text{Aff}(\mathcal{O})$ maps (h,c) to $(d(f)h, \rho(f)c)$ where $d(f) \in SL(\mathcal{O})$ is the linear part of f).

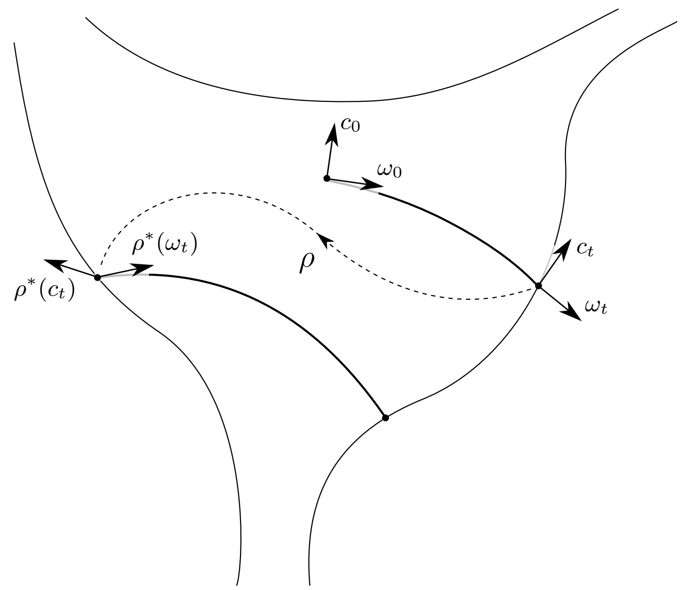

FIGURE 14. Kontsevich–Zorich cocycle over a fundamental domain of $SL(2,\mathbb{R})/SL(\mathcal{O})$.

The curious reader might want to look in the literature for the keywords *local systems*, *Hodge bundle*, *variations of Hodge structures* (of weight 1), *Gauss–Manin connection*, *Kontsevich–Zorich cocycle*, ... for more informations.

Remark 47. In this setting, the Kontsevich–Zorich cocycle over the *Teichmüller flow* is the action of $g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on the flat bundle constructed above.

3.3. **Monodromy constraints.** Let $\mathcal{O}$ be an origami and denote by $G = \text{Aut}(\mathcal{O})$ its finite group of automorphisms. In the sequel, we will use the representation theory of the finite group G to impose restrictions on the natural representation $\rho : \text{Aff}(\mathcal{O}) \to Sp(H_1(\mathcal{O},\mathbb{Z}))$.

Observe that $\text{Aff}(\mathcal{O})$ acts on G by conjugation because the conjugate of a translation by an affine map is a translation. In other words, we have a natural homomorphism $\text{Aff}(\mathcal{O}) \to \text{Sym}(G)$.

Hence, the kernel $\text{Aff}_*(\mathcal{O})$ of $\text{Aff}(\mathcal{O}) \to \text{Sym}(G)$ is a *finite-index* subgroup of $\text{Aff}_*(\mathcal{O})$ such that any $f \in \text{Aff}_*(\mathcal{O})$ *commutes* with all $g \in G$.

In particular, if we decompose the G-module $H_1(\mathcal{O},\mathbb{R})$ into *G-isotypical components* W_α, i.e.,

$$H_1(\mathcal{O},\mathbb{R}) = \bigoplus_{\alpha\in\mathrm{Irr}_{\mathbb{R}}(G)} W_\alpha$$

where $\mathrm{Irr}_{\mathbb{R}}(G)$ is the set of irreducible G-representations, then the matrices in $\rho(\mathrm{Aff}_*(\mathcal{O}))$ *preserve* each W_α.

It is known (cf. Serre's book [21]) that there are three types of $\alpha\in\mathrm{Irr}_{\mathbb{R}}(G)$ depending on its *commuting algebra* D_α:

- α is *real*, i.e., $D_\alpha\simeq\mathbb{R}$;
- α is *complex*, i.e., $D_\alpha\simeq\mathbb{C}$;
- α is *quaternionic*, i.e., $D_\alpha\simeq H=\{a+bi+cj+dk : a,b,c,b\in\mathbb{R}, i^2=j^2=k^2=-1, ij=k\}$.

This information was exploited in [16] to prove that:

- if α is *real*, then $\rho(\mathrm{Aff}_*(\mathcal{O}))|_{W_\alpha}\subset Sp(d_\alpha,\mathbb{R})$ (where $d_\alpha=\dim(W_\alpha)$);
- if α is *complex*, then $\rho(\mathrm{Aff}_*(\mathcal{O}))|_{W_\alpha}\subset SU(p_\alpha,q_\alpha)$;
- α is *quaternionic*, then $\rho(\mathrm{Aff}_*(\mathcal{O}))|_{W_\alpha}\subset SO^*(2n_\alpha)$.[13]

Remark 48. Filip [7] studied recently the constraints on the Kontsevich–Zorich cocycle in more general situations.

3.4. **Lyapunov exponents.** In their study of Lorenz gases (in Statistical Mechanics), P. Ehrenfest and T. Ehrenfest introduced (circa 1920) their *wind-tree models*: for instance, the $\mathbb{Z}^2$-periodic version of Ehrenfest's wind-tree model of Lorenz gases consists of studying billard trajectories on a billiard table $X_{a,b}$ obtained from the plane $\mathbb{R}^2$ by placing rectangular obstacles (whose sides are parallel to the axes) of dimensions $0<a,b<1$ centered at each point of $\mathbb{Z}^2$. See Figure 15 for an illustration.

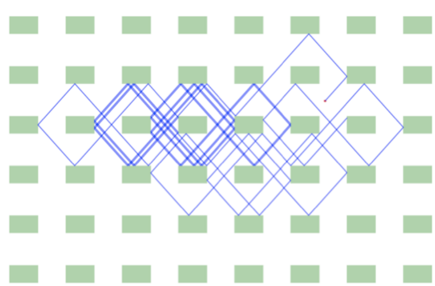

FIGURE 15. $\mathbb{Z}^2$-periodic wind-tree model (after Vincent Delecroix).

Given a point $x\in X_{a,b}$ and a direction θ, we denote by $\{\phi_\theta^t(x)\}_{t\in\mathbb{R}}$ the billiard trajectory in $X_{a,b}$ starting at x in the direction θ. The *diffusion rate* of a billiard trajectory $\{\phi_\theta^t(x)\}_{t\in\mathbb{R}}$ is

$$\limsup_{t\to+\infty}\frac{\log d_{\mathbb{R}^2}(x,\phi_\theta^t(x))}{\log t}$$

[13] Recall that $SO^*(2n):=\{A\in GL(n,H):\sigma(A)A=\mathrm{Id}\}$, where $\sigma(a+bi+cj+dk):=a-bi+cj+dk$.

where $d_{\mathbb{R}^2}$ stands for the Euclidean distance.

Intuitively, the diffusion rates are physically interesting because they measure the polynomial speeds of escape to infinity of billiard trajectories in $X_{a,b}$.

In 1980, Hardy and Weber conjectured that the typical diffusion rate is *abnormal*[14]: for Lebesgue almost every $x \in X_{a,b}$ and θ, one has $\limsup\limits_{t\to+\infty} \frac{\log d_{\mathbb{R}^2}(x,\phi_\theta^t(x))}{\log t} \neq 1/2$.

This conjecture was confirmed in 2014 by Delecroix–Hubert–Lelièvre [1]: they showed that

$$\limsup_{t\to+\infty} \frac{\log d_{\mathbb{R}^2}(x,\phi_\theta^t(x))}{\log t} = 2/3$$

for Lebesgue almost every $x \in X_{a,b}$ and θ.

As it turns out, the sole (currently) known proof of this fact relies on the *Lyapunov exponents* of the Kontsevich–Zorich cocycle.

Remark 49. It is unlikely that the diffusion rate 2/3 above admits a "simple" or "intuitive" explanation. Indeed, Delecroix–Zorich [2] showed in 2015 that the simple variant of the $\mathbb{Z}^2$-periodic wind-tree model where "one obstacle out of four is removed" (see Figure 16) has the complicated typical diffusion rate 491/1053.

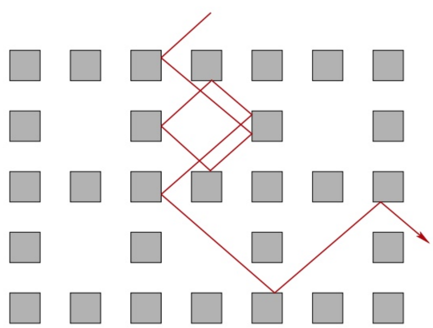

Figure 10. A square-tiled wind-tree where we regularly remove one obstacle in every repetitive pattern of four.

Figure 16. A wind-tree model extracted from Fig. 10 in Delecroix–Zorich paper [2].

Intuitively, Lyapunov exponents of a linear cocycle are *relative* versions of eigenvalues of matrices: roughly speaking, linear cocycles are families of matrices and Lyapunov exponents are families of (logarithms of moduli of) eigenvalues.

In the particular context of the Kontsevich–Zorich monodromy $\rho : \mathrm{Aff}(\mathcal{O}) \to Sp(H_1(\mathcal{O},\mathbb{R}))$ of an origami $\mathcal{O}$, the Lyapunov exponents of $\mathcal{O}$ are

$$\lim_{n\to\infty} \frac{\log\|\rho(A_n\dots A_1)v\|}{\log\|\rho(A_n\dots A_1)\|}, \quad v \in H_1(\mathcal{O},\mathbb{R})$$

[14]This nomenclature is motivated by the fact that random walks and Sinai billiards satisfy the so-called *central limit theorem* saying that the corresponding trajectories are driven by the *normal* distribution and, in particular, the typical diffusion rate is 1/2.

where $A_n \dots A_1$ is a *random product* of elements of $\mathrm{Aff}(\mathcal{O})$ with respect to an appropriate probability measure with full support.

Remark 50. This is *not* the usual definition of Lyapunov exponents of $\mathcal{O}$: normally, we consider the elements of $\mathrm{Aff}(\mathcal{O})$ appearing along a typical orbit of the Teichmüller flow $g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on $SL(2,\mathbb{R})/SL(\mathcal{O})$. The fact that these definitions coincide relies on the important works of Furstenberg, Margulis, Kaimanovich, ... saying that a typical g_t-orbit is "tracked" by an appropriate random walk: see [5] and the references therein for more details.

Since $\rho(\mathrm{Aff}(\mathcal{O}))$ is a subgroup of the group $Sp(H_1(\mathcal{O},\mathbb{R}))$ of symplectic automorphisms of the $2g$-dimensional space $H_1(\mathcal{O},\mathbb{R})$, it is not hard to check that the Lyapunov exponents of $\mathcal{O}$ have the form[15]

$$1 = \lambda_1(\mathcal{O}) \geq \lambda_2(\mathcal{O}) \geq \dots \geq \lambda_g(\mathcal{O}) \geq -\lambda_g(\mathcal{O}) \geq \dots \geq -\lambda_2(\mathcal{O}) \geq -\lambda_1(\mathcal{O}) = -1$$

Moreover, it was shown by Forni [8] that $\lambda_2(\mathcal{O}) < 1$, but it is not easy in general to compute the values of Lyapunov exponents[16].

Nevertheless, Eskin–Kontsevich–Zorich [4] proved that the *sum* $\lambda_1(\mathcal{O}) + \dots + \lambda_g(\mathcal{O})$ are given by the following *explicit* formula:

$$1 + \lambda_2(\mathcal{O}) + \dots + \lambda_g(\mathcal{O}) = \frac{1}{12}\sum_{n=1}^{\sigma} \frac{k_n(k_n+2)}{k_n+1} + \frac{1}{\# SL(2,\mathbb{Z})\mathcal{O}} \sum_{\substack{X \in SL(2,\mathbb{Z})\mathcal{O}, \\ c \text{ cycle of } h_X}} \frac{1}{\text{length of } c},$$

where an origami X in the $SL(2,\mathbb{Z})$-orbit of $\mathcal{O}$ in the stratum $\mathcal{H}(k_1,\dots,k_\sigma)$ is thought as a pair of permutations (h_X, v_X).

Remark 51. This combinatorial formula is well-adapted for computer experiments (in Sage, say).

The proof of the Eskin–Kontsevich–Zorich formula is out of the scope of these notes: it is a long argument (the original article [4] has more than 100 pages) using heavy technology from algebraic geometry, hyperbolic geometry, etc.

For this reason, we will close our discussions with the following application of Eskin–Kontsevich–Zorich formula to the calculation of the Lyapunov exponents of the Eierlegende Wollmilchsau EW (from Example 8 and Figure 10).

In this direction, we observe that EW is an origami of genus 3 in the stratum $\mathcal{H}(1,1,1,1)$, its $SL(2,\mathbb{Z})$-orbit is $SL(2,\mathbb{Z})\cdot EW = \{EW\}$, and the horizontal permutation h_{EW} of EW consists of two cycles of lengths 4. By plugging these informations into Eskin–Kontsevich–Zorich formula, we obtain

$$1 + \lambda_2(EW) + \lambda_3(EW) = \frac{1}{12} \times 4 \times \frac{1\times 3}{2} + \frac{1}{1} \times 2 \times \frac{1}{4} = 1$$

[15]This reflects the fact that the eigenvalues of symplectic matrices come into pairs θ, $1/\theta$.

[16]Nevertheless, one can usually get *numerical* approximations by calculating (with Sage for example) the actions on homology of the matrices $S^{b_n}T^{a_n}\dots S^{b_1}T^{a_1} \in SL(2,\mathbb{Z})$ for random choices of $a_k, b_k \in \mathbb{N}$.

Thus, $\lambda_2(EW) + \lambda_3(EW) = 0$, i.e., $\lambda_2(EW) = \lambda_3(EW) = 0$.

Remark 52. This behavior of Lyapunov exponents of the Eierlegende Wollmilchsau is in sharp contrast with Forni's theorem [8] ensuring that the Lyapunov exponents of the Masur–Veech measures never vanish.

Remark 53. The Lyapunov exponents of the Eierlegende Wollmilchsau can be computed by alternative methods: see, e.g., [9] and [15].

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