1. IETs that do not satisfy Keane's condition.

\[ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, \quad a_1 = a_4, \quad a_3 \neq 0 \]

\[
\begin{align*}
x_1 &= a_1 \\ x_2 &= a_1 + a_2 \\ x_3 &= a_1 + a_2 + a_3 \\
T_{\Pi_2}(x) &= x \\ x \in [x_2, x_3]
\end{align*}
\]

IET is not minimal

\[ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad a = (a_1, a_2, a_3, a_4) \\
(a_1, a_2) \text{ are incommensurable} \]

\[ \Rightarrow T_{\Pi_2} \text{ is minimal and uniquely ergodic.} \]

So, the question either "non-Keane" IET is minimal or not is non-trivial.

2. \( \mathsf{SAF}(T) = \sum \lambda_i \wedge t_i \in \mathbb{R} \wedge \mathbb{R} \)

Rotation: \( \mathsf{SAF}(R_\alpha) = 1 \wedge \alpha = 0 \) \( \Rightarrow \alpha \in \mathbb{Q} \)

In genus 2: \( T \) is ergodic \( \Rightarrow \mathsf{SAF}(T) = 0. \)

\( \mathsf{SAF} \) is a morphism \( \mathcal{G}_{\mathsf{IET}} \rightarrow \mathbb{R} \wedge \mathbb{Q} \wedge \mathbb{R} \).

\( \mathsf{SAF} = 0 \) in general is not an indicator of the non-minimal family of IETs.

3. Theorem (Arnoux-Yoccoz example)

\[ f = \alpha^3 + \alpha^2 + \alpha \]

\[ \xrightarrow{\text{permutation of 2 halves} \oplus \text{translation on } T_i} \]

The 1st return map on \( I_1 \cup I_1 \) is \( T_{\mathsf{AY}} \) (up to rescale). \( T_{\mathsf{AY}} \) are self-similar.

Minimal & uniquely ergodic

It is a horizontal part of a pseudo-Anosov map on a genus 2 surface.

Dilation coefficient = \( \sqrt{2} \).
Symbolic coding:

1 → 1
2 → 2

Substitution

T_{XY}:

\begin{align*}
I & \mapsto I \\
C & \mapsto \sum \\
\Sigma & \mapsto \Sigma^n
\end{align*}

C: coding is "onto" except for a countable set.

C \circ \Sigma^n \Rightarrow \text{has fixed point} (\text{un})

Arnaud - Rauzy words.

\begin{align*}
M_1 &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
M_2 &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
M_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{align*}

Complexity 2n+1.

Episturmian words.

(\Rightarrow \text{always uniquely ergodic})

Definition of the Rauzy gasket.

\[ \Lambda = \{ (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_{>0} : \lambda_1 + \lambda_2 + \lambda_3 = 1 \} \]

Linear maps \( \mathbb{R}^3_{>0} \rightarrow \mathbb{R}^3_{>0} \) defined by \( M_1, M_2, M_3 \):

\[ f_1(\lambda_1, \lambda_2, \lambda_3) = \left( \frac{\lambda_1}{1 + \lambda_2 + \lambda_3}, \frac{\lambda_2}{1 + \lambda_2 + \lambda_3}, \frac{\lambda_3}{1 + \lambda_2 + \lambda_3} \right) \]

Respective projective maps from \( \Lambda \) to itself:

\[ f_2(\lambda_1, \lambda_2, \lambda_3) = \left( \frac{\lambda_1}{1 + \lambda_1 + \lambda_3}, \frac{1}{1 + \lambda_1 + \lambda_3}, \frac{\lambda_3}{1 + \lambda_1 + \lambda_3} \right) \]
Def: The Hausdorff gasket is the maximal subset $\Delta$ of $\Delta$ s.t.
$$R = f_1(R) \cup f_2(R) \cup f_3(R).$$

For $R \subset \mathbb{R}$ infinite sequence $(i_1, i_2, i_3, \ldots)$ of $1, 2, 3$ s.t. $\forall k \in \mathbb{N}$ $f_k(i_1, i_2, i_3, \ldots) \subset \Delta$.

Sequence associated with $\lambda$.
$$\Delta_\lambda = \{ (x_1, x_2, x_3) : x_i \in \{0, 1 \} \}$$

$$f_1(\Delta) \cup f_2(\Delta) \cup f_3(\Delta) = \Delta \Delta_\lambda.$$

Topologically the Hausdorff gasket is equivalent to the Sierpinski gasket but geometry is quite different.

Totally irrational points of $R = \operatorname{Rirr}$

$$R \setminus \operatorname{Rirr} = \bigcup f_i(\partial \Delta).$$

A sequence associated with $i$ is one to one map from $\operatorname{Rirr}$ to the set of all infinite sequences with entries $\{1, 2, 3\}$ in which each symbol of $1, 2, 3$ appears infinitely often.

Thm: $\operatorname{Leb} R = 0$.

(Levitt-[Yoccoz], Aymamov - De Leo, Arneux - Starosty)

Thm: $\operatorname{Hdim} R < 2$.

Question: find a natural measure.
Construction of the flow

Proof function \( r : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \)

\[ r(\lambda, x) = -\log \max_\xi x_\xi \]

\[ X = (\mathbb{R} \times \mathbb{R}) \times (\lambda, x + r(x)) \]

\[ N (x, \mathbb{R}) \times (\lambda, x). \]

\[ \Phi_t(x) = (x, 0) \quad 0 \leq t < r(x) \]

\[ \Phi(x) = (T(x), 0) \]

\[ \frac{1}{x} \left( \frac{(X, T)}{(T(x), 0)} \right) \]

**Theorem:** There exists a measure of maximal entropy for the flow.

Based on the work of Sarig and AGY, Auffelov-Gurevich, Hämemastadt: for the case of Teichmüller flow, this measure is unique Lebesgue.

**6. Entropy of the partition**

\( \mathcal{B} \) - \( \sigma \)-algebra

Partition - disjoint collection of elements of \( \mathcal{B} \) whose union is equal to \( X \).

\[ P_i = \{ A_i \} \quad A_i \neq \emptyset \]

\[ H(P_i) = -\sum \mu(A_i) \log (\mu(A_i)) \]

Entropy
Entropy of measure preserving transformation

\[ P_i = \{ A_1, \ldots, A_n \} \]

\[ \bigvee T^{-i} P_i = \bigoplus_{i,j} \mathbb{A}_{i,j} \] (\( i, j \) c \( \mathbb{A} \) - \( n^q \))

Entropy w.r.t. partition

\[ h(T, P_i) = \lim_{n \to \infty} \frac{1}{n} H \left( \bigvee_{i=0}^{n-1} T^{-i} P_i \right) \]

Entropy of cont. map \( h(\mu) = \sup \{ h(T, P) : P \text{ finite partition of } X \} \)

Topological pressure

\[ P_T(\Phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\bigcup_{i=0}^{h-1} \Phi(T^i x)} \exp \left( \frac{1}{n} \sum_{i=0}^{h-1} \Phi(T^i x) \right) \]

\[ P_T(\Phi) \] quant. exp. growth of periodic orbits

Variational principle

\[ P_T(\Phi) = \sup \{ h(\mu) + \int \Phi \, d\mu : \mu \text{ ergodic} \} \]
Systems of Isometries


- GLP 1994
- notion of orbits
- Rauzy induction definition
- Rauzy gasket in terms of systems of isometries
- Novikov's problem.

Novikov: $M$ closed homologous to zero surface smoothly embedded into $\mathbb{R}^3$.

$H \neq 0$ - vector $H = (H_1, H_2, H_3)$

$\eta$ - 1-form on $\mathbb{R}^3$. $\eta = H_1 dx_1 + H_2 dx_2 + H_3 dx_3$

$\omega = \eta \mid_M$

defines a foliation (defined up to $2\pi k$)

whose leaves lifted to $\mathbb{R}^3$ may or may not be open and have asymptotic directions.

$\int_{c} \omega = 0,$ $c \in H_1(M, \mathbb{Z})$ has zero image in $H_1(\mathbb{R}^3, \mathbb{Z})$ under the embedding.

Suspension complex.

Double suspension surface.

$\psi_i : [a_i, b_i] \rightarrow [c_i, d_i], i = 1, \ldots, k$. $y_i - y_{i+1}$ 2k-tuple of pairwise distinct points of $(0, 1)$.

$\varepsilon > 0$ $[y_i, y_i + \varepsilon]$ pairwise disjoint $c \subseteq [0, 1)$

$D = [0, 1] \times [0, 1] \setminus \{ (a_i, b_i) \times (y_i, y_i + \varepsilon) \cup (c_i, d_i) \times (y_i, y_i + \varepsilon) \}$
A = \{1, 2, \ldots, d\}

substitution over alphabet

endomorphism of the free monoid

\|w\| = \# of letters in w

\|w_j\| = \# of occurrences of letter j in w

AR words: all its factors occur infinitely often

\(A_n\) we have \((d-1)n + 1\) factors of length \(n\) with exactly one left special or right special factor of length \(n\).

AR substitutions: \(\sigma_i : i \rightarrow i\), \(j \in A \setminus \{i\}\)

\(j \rightarrow j^2\)

Thm: The word is AR \(\iff\) its set of factors coincides with the set of factors of a sequence of the form

\[\lim_{n \to \infty} \sigma_{i_n} \cdots \sigma_{i_1}(1)\]

where \((i_n)_{n \geq 0} \in A^\infty\) is s.t. every letter in \(A\) occur infinitely often in \((i_n)_{n \geq 0}\)

Moreover: \((i_n)_{n \geq 0}\) is uniquely defined for given \(w\).
\[ [0,1] \times 10 \approx [0,1] \times 1 \]
\[ [a, b] \times \{ y_i \} \leftrightarrow [a, d_i] \times \{ y_i + \epsilon \} \]
\[ [a, b] \times \{ y_i + \epsilon \} \leftrightarrow [a, d_i] \times \{ y_i + \epsilon \} \]

Collapsing the point every straight line segment of the boundary of \( D \).
\( \text{Thm (AHS)} \quad \text{Hdim (RG)} \leq 2 \)
\( \text{Hdim (RG)} = 1.7 \)

Idea about invariant measure:
\[
\phi_t(x) = (x, t) \quad 0 \leq t < r(x) \\
\phi_{t_1} \circ \phi_{t_2}(x) = (T(x), 0) \\
\text{suspension flow.} \\
(\lambda, t + r(\lambda)) \sim (T(\lambda), t).
\]
\[
\gamma = -\log \max \lambda_i.
\]

**Thm:** there exists a measure of maximal entropy for the flow.

**Systems of isometries.**

Novikov's problem.
Words

Infinite words of complexity \( P(n) = 2n + 1 \) s.t.

- \( n \) is the only one left special factor
- right special factor of length \( n \).
- Left special: \( \exists 2 \) distinct letters \( a \) and \( b \) s.t. \( aw \) and \( bw \) are factors of \( u \).

Prop. \( u \in \text{AR} \)
- \( \sigma \) substitution\( \sigma_i^j(j) = \{ i \} \) if \( j \neq i \) \( \Rightarrow \sigma^i(u) \in \text{AR} \)

Panyu Gasket

\[ \Delta = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \} \]

\( L_1, L_2, L_3 \) - linear maps defined by

\[ M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \]

\( f_i \) - projectivization of \( L_i \)

\[ f_1(x, y, z) = \left( \frac{x}{1 + x_2 + x_3}, \frac{y}{1 + x_2 + x_3}, \frac{z}{1 + x_2 + x_3} \right), \quad f_2(x, y, z) = \left( \frac{x}{1 + x_1 + x_3}, \frac{y}{1 + x_1 + x_3}, \frac{z}{1 + x_1 + x_3} \right), \quad f_3(x, y, z) = \left( \frac{x}{1 + x_1 + x_2}, \frac{y}{1 + x_1 + x_2}, \frac{z}{1 + x_1 + x_2} \right) \]

\[ R \subset \text{maximal subset of } \Delta \]

\[ R = f_1(R) \cup f_2(R) \cup f_3(R) \]