Lecture for 14/06

① IET that do not satisfy Keane's condition.

Ex $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ $\quad a_1 = a_4 \quad a_3 \neq 0$

$x_1 = a_1$
$x_2 = a_1 + a_2$
$x_3 = a_1 + a_2 + a_3$

$T_{\pi,a}(x) = x$
$x \in [x_2, x_3)$
$\Downarrow$
IET is not minimal

Ex $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ $\quad a = (a_1, a_2, a_1, a_2)$

(a_1, a_2) are incommensurable

$\Rightarrow T_{\pi,a}$ is minimal and uniquely ergodic.

So, the question either "non-Keane" IET is minimal or not is non-trivial.

② $SAF(T) = \sum \lambda_i \wedge t_i \in \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$

Rotation: $SAF(R_\alpha) = 1 \wedge_{\mathbb{Q}} \alpha = 0 \Leftrightarrow \alpha \in \mathbb{Q}$

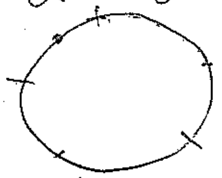
In genus 2: T is ergodic $\Rightarrow SAF(T) \neq 0$.

SAF is a morphism $\mathcal{G}_{IET} \to \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$.

SAF $= 0$ in general is not an indicator of the non-minimal family of IETs.

③ Thm (Arnoux - Yoccoz example)

$1 = \alpha^3 + \alpha^2 + \alpha$.



$\longrightarrow$ permutation of 2 halves + translation on $\mathbb{T}$.

The 1^{st} return map on $I_1 \cup J_1$ is T_{AY} (up to rescale). T_{AY} are self-similar.

Minimal & uniquely ergodic.

It is a horizontal part of a pseudo-Anosov map on a genus 3 surface.

Dilatation coeff. $= 1/\alpha$

Symbolic coding:

$$\begin{array}{l} 1 \\ 2 \\ 3 \end{array} \qquad \sigma: \begin{array}{l} 1 \to 1 \\ 2 \to 21 \\ 1 \to 31 \end{array}$$ substitution

$$\begin{tikzcd} I \arrow[r, "T_{AY}"] \arrow[d, "C"'] & I \arrow[d] \\ \Sigma \arrow[r, "\sigma"] & \Sigma \end{tikzcd}$$

σ.

C: coding

C is "onto" except for a countable set.

$\sigma \quad \Sigma^{\mathbb{N}} \to \Sigma^{\mathbb{N}} \Rightarrow$ has ! fixed point (u_n).

contractive w.r.t. standard metric

Arnoux-Rauzy words.

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Complexity $2n+1$.

Epistarmian words.

($\Rightarrow$ always uniquely ergodic).

④ Definition of the Rauzy gasket.

$$\triangle = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_{\geq 0} : \lambda_1 + \lambda_2 + \lambda_3 = 1\}.$$

L_1, L_2, L_3 linear maps $\mathbb{R}^3_{\geq 0} \to \mathbb{R}^3_{\geq 0}$ defined by M_1, M_2, M_3.

$$f_1(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{1}{1+\lambda_2+\lambda_3}, \frac{\lambda_2}{1+\lambda_2+\lambda_3}, \frac{\lambda_3}{1+\lambda_2+\lambda_3} \right)$$

respective projective maps from $\triangle$ to itself

$$f_2(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{\lambda_1}{1+\lambda_1+\lambda_3}, \frac{1}{1+\lambda_1+\lambda_3}, \frac{\lambda_3}{1+\lambda_1+\lambda_3} \right)$$

Def: The Rauzy gasket is the maximal subset R of $\triangle$ s.t.

$$R = f_1(R) \cup f_2(R) \cup f_3(R).$$

$\lambda \in R \iff \exists$ infinite sequence $(i_1, i_2, i_3 \dots) \in \{1,2,3\}^{\mathbb{N}}$ s.t. $\forall k \in \mathbb{N}$ $\lambda \in f_{(i_1 \dots i_k)}(\triangle)$.

sequence associated with λ.

$\triangle_0 = \{(\lambda_1, \lambda_2, \lambda_3) : \lambda_i \leq 1/2\}$ subset of $\triangle$ consisting of points that satisfy the triangle inequalities

$$f_1(\triangle) \cup f_2(\triangle) \cup f_3(\triangle) = \triangle \setminus \triangle_0$$

Topologically the Rauzy gasket is equivalent to Sierpinski gasket but geometry is quite different.

Totally irrational points of $R = R_{irr}$

$R \setminus R_{irr} = \bigcup f_i(\partial\triangle)$. $i = \{1,2,3\}$.

$\lambda \mapsto$ sequence associated with λ is one to one map from R_{irr} to the set of all infinite sequences with entries $\{1,2,3\}$ in which each symbol of $\{1,2,3\}$ appears infinitely often.

Thm $\quad \mathrm{Leb}\, R = 0$.

(Levitt - [Yocco7], Dynnikov - De Leo, Arnoux - Starosta)

Thm $\quad \mathrm{Hdim}\, R < 2$.

Question: find a natural measure

5 Construction of the flow

Roof function $r: R_{irr} \times \Delta_{irr} \to \mathbb{R}$

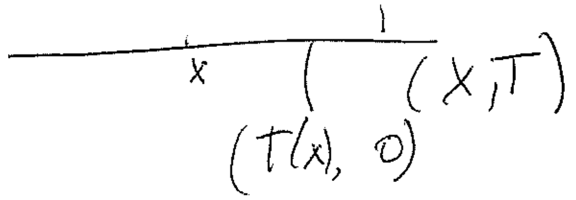
$$r(\lambda, \xi) = -\log \max_i \lambda_i$$

$$X = \left(R_{irr} \cancel{\times \Delta_{irr}} \right) \times \mathbb{R} \big/ (\lambda, a + r(\lambda)) \sim$$

$$\sim (F(\lambda), a).$$

$$\varphi_t(x) = (x, t) \qquad 0 \le t < r(x)$$

$$\varphi_{r(x)}(x) = (T(x), 0).$$



<u>Thm</u>: there exists a measure of maximal entropy for the flow

Based on the work of Sarig and AGY.

Bufetov - Gurevich, Hämenstadt: for the case of Teichmüller flow

this measure is ~~uniq~~ Lebesgue.

6 $\boxed{\text{Entropy of the partition}}$

$\mathcal{B}$ - σ-algebra

Partition of X - disjoint collection of elements of $\mathcal{B}$ whose union is equal to X.

$$P_1 = \{A_1 \ldots A_n\}$$

$$H(P_1) = -\sum \mu(A_i) \log (A_i).$$

entropy .

Entropy of measure preserving transformation
$P_1 = \{A_1 \ldots A_n\}$

$$\bigvee T^{-i} P_1 = \{\bigcap T^{-i} A_{i_j} \; . \; i_j \in \{1 \ldots n\}\}$$

Entropy w.r.t. partition $h(T, P_1) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i} P_1\right).$

Entropy of cont. map of the compact space: $h(\mu) = \sup\{h(T, P) : P$ finite partition of $X\}$.

Topological pressure:

$$P_T(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{T^n x = x} \exp\left(\sum_{i=0}^{n-1} \phi(T^i x)\right).$$

$P_T(0)$ quant. exp. growth of periodic orbits

Variational principle $P_T(\phi) = \sup\{h(\mu) + \int \phi \, d\mu : \mu \in M_T\}$

Systems of isometries

- GLP 1994
  notion of orbits
- Rauzy induction definition
- Rauzy gasket in terms of systems of isometries
- Novikov's problem.

Novikov: M closed homologous to zero surface smoothly embedded into $\mathbb{T}^3$

$H \neq 0$ - vector $\quad H = (H_1, H_2, H_3)$

η - 1-form on $\mathbb{T}^3$. $\quad \eta = H_1 dx_1 + H_2 dx_2 + H_3 dx_3$

angular coordinates (defined up to $2\pi k$)

$\omega = \eta|_M$

defines a foliation whose leaves lifted to $\mathbb{R}^3$ may or may not be open and have asymptotic directions

$\int_c \omega = 0 \quad c \in H_1(M, \mathbb{Z})$ has zero image in $H_1(\mathbb{T}^3, \mathbb{Z})$ under the embedding $M \hookrightarrow \mathbb{T}^3$

Suspension complex.

Double suspension surface.

$\psi_i \ \{[a_i, b_i] \to [c_i, d_i] \ i = 1 .. k\}$

$(y_1 .. y_{2k})$ $2k$-tuple of pairwise distinct points of $(0,1)$

$\varepsilon > 0 \quad [y_i, y_i + \varepsilon]$ pairwise disjoint $\subset [0,1)$

$D = [0,1] \times [0,1] \setminus \left((a_i, b_i) \times (y_i, y_i + \varepsilon) \cup (c_i, d_i) \times (y_{i+k}, y_{i+k} + \varepsilon)\right)$

Arnoux - Rauzy words

$\mathcal{A} = \{1, 2, \ldots, d\}$

substitution over alphabet — endomorphism of the free monoid

w — $|w| = $ # of letters in w — $|w_j|$ - # of occurrencies of letter j in w

AR words: all its factors occur infinitely often
$\forall n$ we have $(d-1)n+1$ factors of length n with exactly one left special & right special factor of length n.

AR substitutions: $\sigma_i : i \mapsto i$
$j \longmapsto ji \quad j \in \mathcal{A} \setminus \{i\}$

<u>Thm</u>: The word is AR $\iff$ its set of factors coincides with the set of factors of a sequence of the form $\lim\limits_{n \to \infty} \sigma_{i_0} \ldots \sigma_{i_n}(1)$, where $(i_n)_{n \geq 0} \in \mathcal{A}^{\mathbb{N}}$ is s.t. every letter in $\mathcal{A}$ occur infinitely often in $(i_n)_{n \geq 0}$

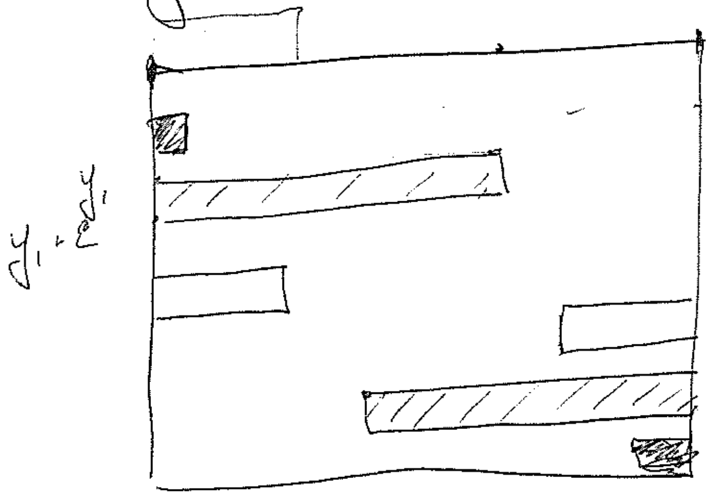
Moreover: $(i_n)_{n \geq 0}$ is uniquely defined for given ω.

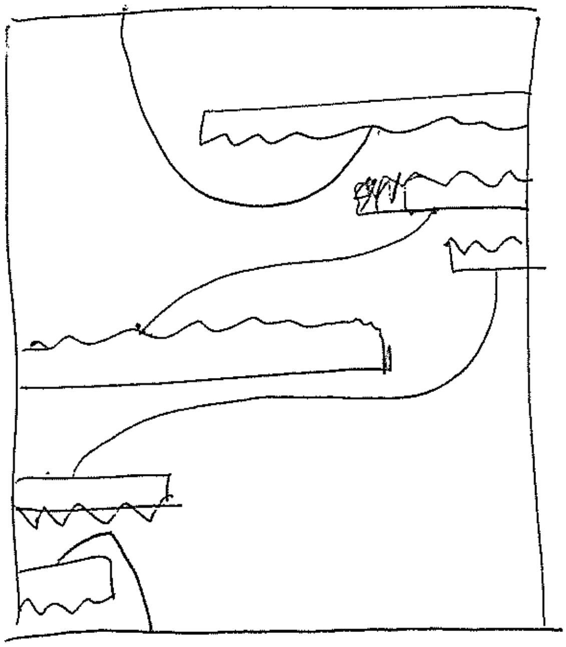
$$[0,1]\times\{0\} \sim [0,1]\times\{1\}$$

$$[a_i,b_i]\times\{y_i\} \longleftrightarrow [c_i,d_i]\times\{y_{i+k}+\varepsilon\}.$$

$$[a_i,b_i]\times\{y_i+\varepsilon\} \longleftrightarrow [c_i,d_i]\times\{y_{i+k}\}.$$

collapsing ~~every~~ to a point every straight line segment of the boundary of D.





Thm (AHS) $\quad \mathrm{Hdim}(RG) < 2$

Numerics: $\quad \mathrm{Hdim}(RG) = 1.7$

Idea about invariant measure:
$$\Phi_t(x) = (x, t) \qquad 0 \le t < r(x)$$
$$\Phi_{r(x)}(x) = (T(x), 0)$$

suspension flow.

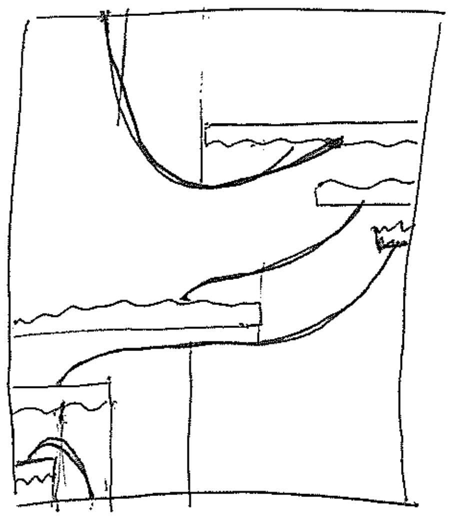
$$(\lambda, t + r(\lambda)) \sim (T(\lambda), t).$$
$$r = -\log \max_i \lambda_i .$$

Thm: there exists a measure of maximal entropy for the flow

Systems of isometries.

~~Novikov's problem:~~

~~M~~



Novikov's problem

◦ A R words

infinite words of complexity $\mathcal{C}(n) = 2n+1$ s.t.
$\forall n$ there is only one left special factor
& right special factor of length n.

Left special: $\exists$ 2 distinct letters a and b
s.t. aw and bw
are factors of u.

Prop $u \in AR$
σ. substitution $\Rightarrow \sigma(u) \in AR$

$$\sigma_i(j) = \begin{cases} ij & j \neq i \\ i & j = i \end{cases}$$

Rauzy Gasket

$$\Delta = \{ (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_{\geq 0} : \lambda_1 + \lambda_2 + \lambda_3 = 1 \}$$

L_1, L_2, L_3 - linear maps defined by

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

f_i - projectivization of L_i.

$$f_1(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{1}{1+\lambda_2+\lambda_3}, \frac{\lambda_2}{1+\lambda_2+\lambda_3}, \frac{\lambda_3}{1+\lambda_2+\lambda_3} \right)$$

$$f_2(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{\lambda_1}{1+\lambda_1+\lambda_3}, \frac{1}{1+\lambda_1+\lambda_3}, \frac{\lambda_3}{1+\lambda_1+\lambda_3} \right)$$

$$f_3(\lambda_1, \lambda_2, \lambda_3) = \left(\frac{\lambda_1}{1+\lambda_1+\lambda_2}, \frac{\lambda_2}{1+\lambda_1+\lambda_2}, \frac{1}{1+\lambda_1+\lambda_2} \right)$$

$f_{i_1} \circ \dots \circ f_{i_k}$

RG $\underset{\text{maximal subset of } \Delta \text{ s.t.}}{\mathcal{R}} \overset{i}{=} f_1(\mathcal{R}) \cup f_2(\mathcal{R}) \cup f_3(\mathcal{R})$