$$G = SL_2(\mathbb{R}), \quad r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad g_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

$$u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \quad U = \{u_s : s \in \mathbb{R}\}.$$

"horocycle flow"

How to solve a counting problem on a polygon P:

$$P \overset{\text{unfolding}}{\rightsquigarrow} M_P \overset{\text{renormalization}}{\rightsquigarrow} \text{equidistribution problem}$$

polygon ↑ trans. surf. ↑

$$\frac{1}{2\pi} \left| \{\theta \in [0, 2\pi] : g_t r_\theta M_P \in \mathcal{C}(\varepsilon)\} \right| \xrightarrow[t\to\infty]{?} \frac{\mathrm{Vol}(\mathcal{C}(\varepsilon))}{\mathrm{Vol}(\text{moduli space})}$$

In general, given M, want to:

① Understand $\overline{GM}$, equip it with the structure of a moduli space + natural measure Vol

(McMullen $g=2$, Eskin-Mirzakhani-Mohammadi $g \geq 2$)

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2 Define measures ν_t on $\overline{GM}$ by

$$\nu_t(A) = \frac{1}{2\pi} \left| \{\theta \in [0, 2\pi] : g_t r_\theta M \in A\} \right|$$

and prove $\nu_t \xrightarrow[t\to\infty]{} \mathrm{Vol}_{\overline{GM}}$ (w.r.t. weak-* top on measures).

Ex. 1 Any weak-* limit of ν_t is U-invariant.

Hint: Let $u = u_s \in \mathcal{U}$, define θ_0 by

$$e^t \sin\theta_0 = -s,$$ and g_0 by

$$u_s g_t = g_0 g_t r_{\theta_0}.$$

(g_0 depends on s and t). Then $g_0 \xrightarrow[t\to\infty]{} Id$

So $u_s g_t r_\theta \approx g_t r_{\theta_0 + \theta}$.

Upshot: Want to describe invariant measures and orbit-closures for both G and U, actions.

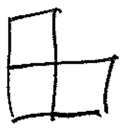
Examples for G-action

(G1) Strata. $\mathcal{H}_1(a_1, \dots, a_n) = \mathcal{H}$

Each $\mathcal{H}$ is equipped with a natural finite G-inv. measure. (Masur, Veech, Masur-Smillie)
This measure is ergodic for g_t-action (Masur) hence for G.

$\Rightarrow$ a.e. G-orbit is dense.

(G2) M a square tiled surface, eg.

$M =$  . $\Gamma = \Gamma_M = \{ g \in G : gM = M \}$
Veech group of M.

Γ_M is of finite index in $SL_2(\mathbb{Z})$, hence a lattice.
A surface with this property is called a Veech surface. Non-arithmetic example: regular $2n$-gon. proof: postponed.

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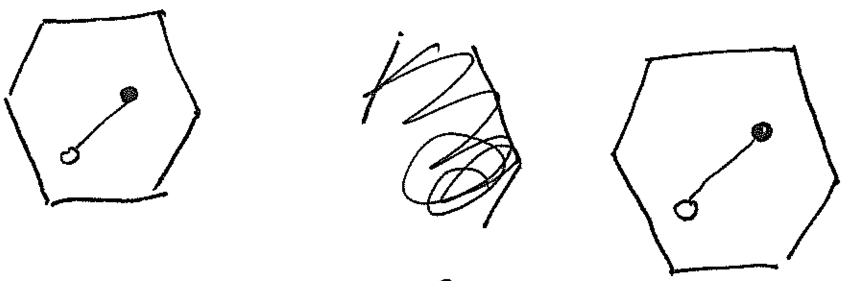
Prop 1: For a Veech surface, GM is closed. $GM \cong G/\Gamma_M$ carries finite G-inv. measure.

Thm If GM is closed then M is a Veech surface.

proofs: postponed.

(G3) $\mathcal{E}_4$-locus. $\mathcal{E}_4 = \{M \in \mathcal{H}(1,1) :$ M is a 2:1 translation cover of a surface in $\mathcal{H}(0,0)$ $\}$.

Picture:



This is a 4:1 cover of $\mathcal{H}(0,0)$, carries a finite measure which is a pullback.

Generalization: $\mathcal{E}_D$, $D \equiv 0$ or $1 \mod 4$ in $\mathcal{H}(1,1)$.

(Calta, McMullen).

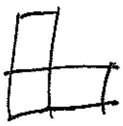
Thm (McMullen '05) That's all in genus 2.

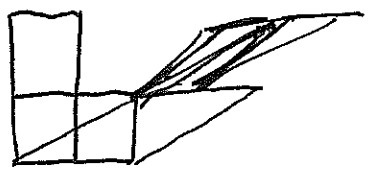
Thm (EMM '14) Similar picture for all $g \geq 3$: Each orbit closure is an orbifold with many structures, specifically is the support of an ergodic G-inv. measure.

Examples for U-action

Prop (Howe-Moore) If G acts ergodically on a finite measure space, restriction of the action to U is also ergodic.

Thus: (U0) All previous G-examples are also U-orbit closures.

(U1) M=  . Mathers explained $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} \in \Gamma_M$.

(pf. by picture: )

So UM periodic.

Are there other U-orbits inside GM?

Thm (Hedlund) on G/Γ, a U-orbit is either periodic or dense.

We will discuss the proof tomorrow.

(12) Generalizing U1, have

<u>Prop 2</u>: Let $M \in \mathcal{H}$. Then UM periodic

$\iff$ M is a union of horizontal cylinders $C_1, \dots, C_m$ with heights h_i, circumferences c_i, and there are $k_i \in \mathbb{N}$ s.t.

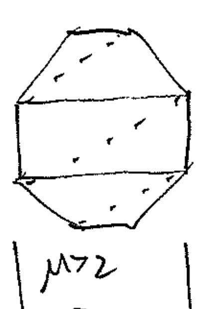
$$k_1 \frac{c_1}{h_1} = k_2 \frac{c_2}{h_2} = \dots = k_m \frac{c_m}{h_m} \overset{\text{def}}{=:} \mu$$

In this case $\begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in \Gamma_M$.

<u>Ex 2</u> Prove $\Leftarrow$

Proof of $\Rightarrow$ postponed.

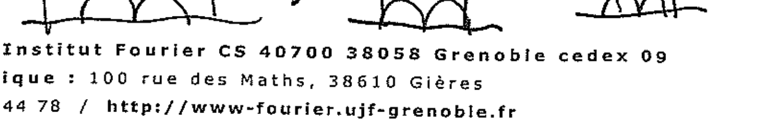
~~Ex~~ Proof that the $2n$-gon is a Veech surface:



Normalize by conjugation so that $u_1 = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$, $|\mu| < 2$.

<u>Ex 3</u> Get two parabolics u_1, u_2

<u>Ex 4</u>



~~Note~~ What happens if we remove this arithmetic condition that the $\frac{c_i}{h_i}$ are commensurable?

<u>Prop</u> M a union of hor. cylinders $C_1, \dots, C_m$ with heights h_i, circumfrences c_i, let $\mu_i = \frac{c_i}{h_i}$.

(U3) Then $\overline{UM} \cong$ torus of dimension

$$d = \dim_{\mathbb{Q}} \operatorname{span}_{\mathbb{Q}} (\mu_i)$$

<u>Pf</u> The geometry of M is determined by:

(i) h_i, c_i. (ii) lengths of horizontal segments in ∂C_i (iii) gluing pattern of the C_i.

(iv) "twists": horizontal component of a segment passing through C_i. ~~(v) gluing pattern~~. Denote by τ_i.

Under U-action, (i), (ii), (iii) unchanged.

τ_i vary in $\mathbb{R}/c_i\mathbb{Z}$ by: $\underset{\text{twist on } M}{\tau_i} \longmapsto \underset{\text{twist on } u_s M}{\tau_i + s h_i}$.

So get a linear flow on $\mathbb{R}/c_1\mathbb{Z} \times \dots \times \mathbb{R}/c_m\mathbb{Z} \cong \mathbb{T}^m$

Closure of a linear flow is a subtorus.

<u>EX. 5</u> Show this torus has dim $d = \dim_{\mathbb{Q}}(\operatorname{span}_{\mathbb{Q}}(\mu_i))$.

<u>Thm</u> ("minimal sets classification")

Any $\overline{UM}$ contains a completely periodic surface, and hence a torus as above.

<u>Pf</u> (postponed).

(14) Suppose M has a nontrivial collection of horizontal saddle connections Ξ. Then this is preserved by U-action, as are ~~depo~~ lengths of element of Ξ, topology of components of $M \setminus \Xi$, their area, etc. One gets a space of surfaces with fixed values of this data.

<u>Problem</u> Construct measures, classify connected components, prove ergodicity on these spaces.

All examples up to now were supports of ergodic measures, and were manifolds/orbifolds. There are less regular orbit-closures, these will be discussed in John's second lecture "spiky fish" (including non-integral Hausdorff dimension).

Proof of Prop 1 (assuming Prop 2)

Orbit map $g\Gamma \longmapsto gM$ cts, 1-1, onto.

$G/\Gamma \longrightarrow \mathcal{H}$

Need to show properness, and then it will be a homeo.

Fact about finite volume quotients of G:

$g_n\Gamma \to \infty$ in $G/\Gamma \iff g_n\Gamma g_n^{-1}$ contains parabolics u_n, $u_n \xrightarrow[n\to\infty]{} Id$.

Note: If $\Gamma = \Gamma_M$ then $g_n\Gamma g_n^{-1} = \Gamma_{g_nM}$.

So $g_n\Gamma \to \infty \Rightarrow \Gamma_{g_nM}$ contains small parabolics $\begin{pmatrix} 1 & \mu_n \\ 0 & 1 \end{pmatrix} = u_n$ (up to rotating axes).
(so u_{g_nM} periodic)

By Prop 2, $\exists\, C_1, \ldots, C_m$, c_i, h_i, k_i (all depend on n) with $\frac{k_i c_i}{h_i} = \mu_i = \mu_i^{(n)} \xrightarrow[n\to\infty]{} 0$.

$$\frac{c_i}{h_i} \le \mu_i \qquad c_i h_i = \text{area}(C_i) \le 1$$

$$\Rightarrow c_i \le \mu_i h_i \le \mu_i \frac{1}{c_i} \Rightarrow c_i^2 \le \mu_i \longrightarrow 0$$

$$\Rightarrow c_i \to 0. \text{ So } g_nM \longrightarrow \infty.$$

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Recap from last time.

Want to show:

~~If $\overline{UM}$ perio~~

1. If GM closed then M is a Veech surface (Smillie's thm).

2. For any surface M, $\overline{UM}$ contains a horizontally periodic surface.

Ex. 6 Direction $\Rightarrow$ in Prop 2 follows from minimal sets classification.

Quantitative Nondivergence for U-action

Thm (Veech, following K-M-S):
There is no M s.t. $u_s M \xrightarrow[s\to\infty]{} \infty$.

Contrast with $g_t M \xrightarrow[t\to\infty]{} \infty$ (relation to Masur's criterion)

Idea Draw all length functions
$s \overset{\ell_\gamma}{\longmapsto}$ length of saddle connection γ on $u_s M$.
$\ell_\gamma : \mathbb{R} \to (0,\infty)$

Many strengthenings

Strengthening: ~~For any~~ $\exists K \subset \mathcal{H}$ compact s.t. for all $M \in \mathcal{H}$, either
(i) M has a horizontal saddle connection.
(ii) $\liminf_{T\to\infty} \frac{1}{T} |\{s \in [0,T] : u_s M \in K\}| \geq \frac{1}{2}$.

Sketch of proof of Smillie's theorem.
Orbit map is proper. So pushing haar get a locally finite U-inv. measure on $\mathcal{H}$.
Hopf ratio ergodic theorem. Suppose

$f_1, f_2 \geq 0$ in $L^1(X, \mu)$, a space with an ergodic flow (not necessarily finite ~~and~~ measure)
measure is σ-finite. $\{u_s\}$ is the flow.

Then for a.e. x

$$\frac{\int_0^T f_1(u_s x)\,ds}{\int_0^T f_2(u_s x)\,ds} \xrightarrow[T\to\infty]{} \frac{\int f_1\,d\mu}{\int f_2\,d\mu}.$$

In our case work with $f_1 = \mathbb{1}_{K_1}$

K_1 a large compact set.
$f_2 = \mathbb{1}_K$, K from Strengthening.

Then for a.e. M

$$2 = \frac{T}{\frac{1}{2}T} \underset{\text{nondivergence}}{\geq} \frac{|\{s\in[0,T] : u_s M \in K_1\}|}{|\{s\in[0,T] : u_s M \in K_2\}|} \xrightarrow[T\to\infty]{\text{Hopf}} \frac{\mu(K_1)}{\mu(K)}$$

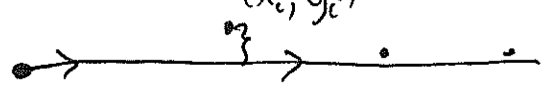
So $\mu(K_1) \leq 2\mu(K)$. This is true for all compact K_1.
So $\mu(\mathcal{H}) \leq 2\mu(K) < \infty$.

Proof of minimal set classification

Induction on ~~complexity of~~ $|\Xi|$.
where Ξ are the horizontal saddle connections on M.
Want to show that any M which is not completely periodic, $\overline{U_s M}$ contains M', $|\Xi_{M'}| > |\Xi_M|$.

Picture.

$x_i \to \infty$
$y_i \to 0$ $\quad y_i \neq 0$



have "near approaches" to singular points.

Applying u_{s_i}, can ensure that $u_{s_i} M = M_i$ has a saddle connection of height y_i and horizontal length $l = x_i + s_i y_i$. M_i has all of Ξ.

Taking a limit set M'.

Problem: what if $M_i \to \infty$?

Fix relax to have $x_i + s_i y_i \in [1, 2]$. Get a long segment. Apply nondivergence ~~to~~ ensure $M_i \in K$.

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Qu: Can we find interesting U-orbits <u>inside</u> closed G-orbits? No, because of:

<u>Thm</u> (Hedlund '36) For any finite volume G/Γ, for any $x = g\Gamma \in G/\Gamma$, Ux is either periodic or dense.

Here is a stronger statement.

<u>Thm</u> (Furstenberg '76, ..., Ratner)

Any ergodic U-inv measure on G/Γ is either length measure on a periodic orbit or is the (unique) G-inv. measure.

In the remaining time we sketch some elements of the proof.

We start with a "mystery" U-inv. prob. measure μ, which is not supported on a single orbit, and show that it is G-inv.

Let $A = \{g_t : t \in \mathbb{R}\}$.

<u>Main step</u>: Any U-invariant ergodic measure not supported on a single orbit is A-inv.

We will prove a slightly weaker statement: any such measure is g_{t_0}-inv. for some $t_0 \neq 0$.

<u>Def</u> $x \in G/\Gamma$ is <u>generic</u> (for U and μ) if for all $f \in C_c(G/\Gamma)$,

$$\frac{1}{T}\int_0^T f(u_s x)\,ds \xrightarrow[T\to\infty]{} \int_{G/\Gamma} f\,d\mu.$$

<u>Thm</u> (Birkhoff ergodic thm).

μ-a.e. x is generic.

<u>Prop</u> Let $X_0 \subset G/\Gamma$ be the set of generic points, and there is $a \in A \setminus \{Id\}$ s.t. $aX \cap X \neq \emptyset$. Then μ is a-inv.

(ex.)

Proof: enough to show that for all

$f \in C_c(G/\Gamma)$, $\int_{G/\Gamma} f(x)\,d\mu(x) = \int_{G/\Gamma} f(ax)\,d\mu(x)$.

Let $x_0 \in G/\Gamma$ generic so that ax_0 generic.

~~Then~~ Let $f^a(x) = f(ax)$, for $f \in C_c(G/\Gamma)$.

~~Then~~ Want to show $\int f^a\,d\mu = \int f\,d\mu$.

$$\int f^a(x) \xleftarrow[T\to\infty]{x \text{ is gen } \& f^a \in C_c(G/\Gamma)} \frac{1}{T}\int_0^T f(au_s x)\,ds =$$

$$= \frac{1}{T}\int_0^T f(au_s a^{-1} ax)\,ds = \frac{1}{T_1}\int_0^{T_1} f(u_{s_1^*} ax)\,ds_1^* \xrightarrow[T_1\to\infty]{ax \text{ is generic}} \int f\,d\mu$$

$T_1 = e^{t_0} T \xrightarrow[T\to\infty]{} \infty$

$s_1^* = e^{t_0}s$, $a = g_{t_0}$

Let $P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $P^- = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$.

Suppose $g_n \in G$ s.t. $g_n \to Id$, $g_n \notin P$.

Write $g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ $\quad a_n, d_n \to 1$, $b_n \to 0$
$0 \neq c_n \to 0$.

Define for each $s > 0$, $n \in \mathbb{N}$, a function

$$s' = s'(s,n) = \frac{a_n s - b_n}{d_n - c_n s}.$$

This choice is designed so that

$$u_{s'} g_n = \underbrace{\begin{pmatrix} \frac{1}{d_n - c_n s} & 0 \\ c_n & d_n - c_n s \end{pmatrix}}_{P_{n,s} \in P^-} u_s$$

Choose s_n s.t. $\|P_{n,s_n} - Id\| = 1$.

<u>Ex.</u> Then there is $c \in (0,1)$ (independent of n) so that for $s \in [(1-c)s_n, s_n]$ $\|P_{n,s} - Id\| \in [c, 1]$, and any accumulation point of $\{P_{n,s}\}$ $(s \in [(1-c)s_n, s_n])$ is in A.

Picture