MAPPING CLASS GROUPS

Part I. The finite-type case

Let S be a connected, orientable surface with $\pi_1(S)$ fin. gen. Classif. of surfaces tells us that $S \cong S_{g,n}^b$

Comment on punctures vs. marked points.

DEF (Mapping class group)

$\mathrm{Mod}(S) = \{ f : S \to S$ or. pres homeo $\} /$ homotopy ; here, require that homeos/~~hom~~ homotopy preserve ∂S pointwise.

Comment on what MCG does to punctures.

Two related groups:

$\mathrm{Mod}^{\pm}(S)$: the extended mapping class group

$\mathrm{PMod}(S)$: the pure mapping class group

$$1 \longrightarrow \mathrm{PMod}(S) \longrightarrow \mathrm{Mod}(S) \longrightarrow \mathrm{Sym}(n) \longrightarrow 1$$

MCG ↷ Teich.

$\boxed{\text{Royden}}$

1. Examples.

① $\mathrm{Mod}(\mathbb{D}^2) = \{1\}$. Indeed, given $f : \mathbb{D}^2 \longrightarrow \mathbb{D}^2$, do

$$H_{1-t} : \mathbb{D}^2_{1-t} \longrightarrow \mathbb{D}^2_{1-t} \quad \begin{cases} F(x,t) = H_{1-t} f H_{1-t}^{-1} & \text{if } x \in \mathbb{D}(0, 1-t) \\ F(x,t) = x & \text{if } x \in \mathbb{D}^2 - \mathbb{D}(0,1-t). \end{cases}$$

$F(x,1) = x \ \forall x$ gives a homotopy between f and id.

② $\mathrm{Mod}(\mathbb{D}^2 - \{(0,0)\}) = \{1\}$. Same.

③ If $n \leq 3$, then $\mathrm{PMod}(S_{0,n})$ is trivial.

<u>Proof</u>. For $n = 1$ do as in ①.
For $n = 0$, homotope so that homeo fixes a point.
For $n = 2,3$ use arcs. Can ~~dco~~ homotope so that arcs are fixed pointwise.

④. Observe that there is a natural homomorphism

$$\mathrm{Mod}^{\pm}(T^2) \longrightarrow GL(2,\mathbb{Z}) = \mathrm{Out}(\pi_1(S))$$

which is not hard to see is ~~surjec~~ bijective.

Thus $\mathrm{Mod}^{\pm}(T^2) \cong GL(2,\mathbb{Z})$ and $\mathrm{Mod}(T^2) \cong SL(2,\mathbb{Z})$.

Part of (Note: $\mathrm{Mod}(T^2_*) \cong SL(2,\mathbb{Z})$ also)

Theorem (Dehn–Nielsen–Baer): If S is closed then $\mathrm{Mod}^{\pm}(S) \cong \mathrm{Out}(\pi_1(S))$

2. Types of element

① The homotopy class of a periodic homeo. Mention Nielsen Realization.

② Dehn twists. Let $A = [0,1] \times [0,1] / (x,0)\sim(x,1)$

LEFT TWIST.

and $\tau : A \longrightarrow A$ given by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Then given a simple closed curve α, a representative a and a neighborhood N_a ($i : N_a \to A$) do T_α: $i^{-1}\tau i$ on N_a and identity on the rest of S.

Note. $[T_\alpha]$ depends only on $\{\alpha\}$.

Note 2. (Exercise) T_α has infinite order.

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \curvearrowright T$

Example: Anosov map

③ pseudo-Anosov: Leininger's lectures.

④. Nielsen–Thurston classification: a non-reducible element of infinite order ~~has a~~ is pseudo-Anosov.

reducible.

⑤ Normal form: f^n fixes some multicurve and, on the complement, it is periodic or pseudo-Anosov.

3. Relations among twists

(0) $T_\alpha^n = T_\beta^m \iff n = m$ & $\alpha = \beta$.

(1) $[T_\alpha, T_\beta] = 1 \iff i(\alpha,\beta) = 0$.

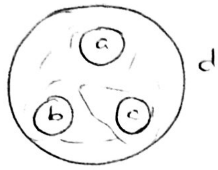
(2) $T_{f(\alpha)} = f T_\alpha f^{-1}$.

(3) $T_\alpha T_\beta T_\alpha = T_\beta T_\alpha T_\beta \iff i(\alpha,\beta) = 1$.

Proof. $\Leftarrow$ $(T_\alpha T_\beta) T_\alpha (T_\beta T_\alpha)^{-1} = T_\beta \iff T_{T_\alpha T_\beta(\alpha)} = T_\beta$.

$\iff T_\alpha T_\beta(\alpha) = \beta$. Check this is the case.

(4) Lantern relation:



$T_a T_b T_c T_d = T_x T_y T_z$.

4. Generation by twists

Theorem (Dehn-Lickorish, Mumford).

If $g \geq 1$ (~~and $n \geq 1$~~) then $PMod(S_{g,n})$ is generated by finitely many twists along nonseparating curves.

$\boxed{\text{Idea of the proof}}$ By induction on g, n.

(1) Induction on n: the Birman short exact sequence:

$$1 \longrightarrow \pi_1(S,*) \xrightarrow{P} PMod(S,*) \longrightarrow PMod(S) \longrightarrow 1.$$

FACT. If α is simple then $p(\alpha) = \alpha_+ \alpha_-^{-1}$.

The base case is $PMod(S_{1,1}) \cong SL(2,\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$

Note: This also proves that $PMod(S_{0,n})$ is generated by twists.
$n \geq 4$

(2) Induction on genus: Let $f \in PMod(S_{g,n})$. Choose α non-separating. If $f(\alpha) = \alpha$, we are done. Also if

$i(\alpha, f(\alpha)) = 1$, for in that case

$T_\alpha T_{f(\alpha)}(\alpha) = f(\alpha)$ by (3)(2)

So we have to prove that there exists a sequence, for arbitrary α, $f(\alpha)$, of curves with

$$\alpha = \beta_0, \beta_1, \ldots, \beta_n = f(\alpha)$$

with $i(\beta_i, \beta_{i+1}) = 1$. One does this □

5. Powell's theorem

No es necesario

Mencionar que por el final

Theorem (Powell, Birman, Mumford) $(\partial S = \phi)$

If $g \geq 3$ then $H_1(\mathrm{PMod}(S), \mathbb{Z}) = \phi$.

Sketch proof.

① $\mathrm{PMod}(S)$ is generated by nonsep twists.

②. 3(2) $\Rightarrow$ nonsep twists are conjugate.

Thus if $\mathrm{PMod}(S) \xrightarrow{\pi} \mathrm{PMod}(S)^{ab}$ then $\pi(\mathrm{PMod}(S)) = \langle A \rangle$

If $g \geq 3$ can find a nonseparating lantern hence $A^4 = A^3$ ∎

Low dimensional cases.

$g=0$. The Birman SES gives

FACT. $\forall n \geq 4$, $\mathrm{PMod}(S_{0,n}) \twoheadrightarrow \mathbb{F}_2$.

$g=1$ $H_1 \cong \mathbb{Z}_{12}$; $g=2$ $H_1 \cong \mathbb{Z}_{10}$

But $\mathrm{PMod}(S) \twoheadrightarrow \mathbb{F}_2$ virtually (use hyperelliptic involution)

Question. Do mapping class groups virtually surject onto free groups?

6. Rigidity

We are going to prove

<u>Theorem (Ivanov - McCarthy)</u>

The natural homomorphism $\mathrm{Mod}^{\pm}(S) \longrightarrow \mathrm{Aut}(\mathrm{Mod}(S))$ is surjective.

A key ingredient is:

<u>Theorem (Bridson)</u>

Let $\phi: \mathrm{Mod}(S) \to \mathrm{Mod}(S')$ be a homomorphism, where genus$(S) \geq 3$. Then ϕ takes Dehn twists to roots of powers of multitwists.

(assume $g \geq 4$)

<u>Proof sketch</u>. Let $\gamma \subset S$ nonseparating, and S_γ the surface (w/ bdy) obtained by removing an open nbd of γ. Consider the inclusion map $i: \mathrm{Mod}(S_\gamma) \to \mathrm{Mod}(S)$ and take $\eta \subset \partial S_\gamma$ with $i(T_\eta) = T_\gamma$. Suppose $\phi(T_\gamma)$ is not a root of a power of a multitwist; ~~set~~ let $Y \subset S'$ be the maximal pseudo-Anosov component of $\phi(T_\gamma)$ and let λ be the union of attracting foliations of $\phi(T_\gamma)$. Then $Z_{\mathrm{Mod}(S')}(\phi(T_\gamma))$ preserves Y and λ. Since we have

$$Z_{\mathrm{Mod}(S_\gamma)}(T_\eta) \xrightarrow{i} Z_{\mathrm{Mod}(S)}(T_\gamma) \longrightarrow Z_{\mathrm{Mod}(S')}(\phi(T_\gamma))$$

we obtain a nontrivial homomorphism $Z_{\mathrm{Mod}(S_\gamma)}(T_\eta) \longrightarrow \mathbb{R}$, which contradicts Powell's theorem ∎

<u>Remark</u>. The same argument gives
"any linear representation of $\mathrm{Mod}(S)$ $(g \geq 3)$ sends twists to roots of unipotents".

Recall we are after:

Theorem (Ivanov). Every automorphism $\text{Mod}(S) \to \text{Mod}(S)$ is
(Algebraic Rigid) induced by a homeomorphism $S \to S$.

[Remark. Known in more generality, e.g. for finite-index subgroups.]

Proof. After Bridson, we know that Dehn twists are sent to roots of powers of multitwists. Using the fact that $\phi: \text{Mod}(S) \to \text{Mod}(S)$ is an automorphism, we deduce that

"Dehn twists are sent to Dehn twists"

So $\phi: \text{Mod}(S) \to \text{Mod}(S)$ induces a map $\phi_*: \mathcal{C}(S) \to \mathcal{C}(S)$ (define)

$$\boxed{\phi(T_\alpha) = T_{\phi_*(\alpha)}}$$

Rmk. The map ϕ_* is a simplicial automorphism.

At this point, we want to prove:

Theorem (Ivanov's Simplicial Rigidity Thm)
($S \neq S_{1,2}$) Every automorphism $\mathcal{C}(S) \to \mathcal{C}(S)$ is induced by a homeo $S \to S$

Rmk. Known for much more general types of maps.

Accepting this for the time being, we finish the proof of Alg. Rig.
By simplicial rigidity we know that there is a homeo $h: S \to S$ with $\phi_*(\alpha) = h(\alpha)$ $\forall \alpha \in \mathcal{C}^{(0)}(S)$. We want to show that

$$\phi(f) = hfh^{-1} \quad \forall f \in \text{Mod}(S).$$

To see this, observe that

$$\phi(fT_\alpha f^{-1}) = \phi(T_{f(\alpha)}) = T_{\phi_*(f(\alpha))} = T_{h(f(\alpha))}$$
$$\|$$
$$\phi(f)\phi(T_\alpha)\phi(f^{-1}) = \phi(f)\, T_{h(\alpha)}\, \phi(f)^{-1} =$$
$$= T_{\phi(f)h(\alpha)}$$

Setting $\alpha = h^{-1}(\beta)$ this means:

$$T_{h(f(h^{-1}(\beta)))} = T_{\phi(f)(\beta)}.$$

Thus, $hfh^{-1}(\beta) = \phi(f)(\beta)$ for every $\beta \in \mathcal{C}^0(S)$ and hence $hfh^{-1} = \phi(f)$ ∎

Let $\varphi: \mathcal{C}(S) \to \mathcal{C}(S)$ be an automorphism. ~~Know: disjoint goes to disjoint & distinct goes to distinct~~. From this its easy to deduce

FACT 1. If α, β fill a small subsurface then $\varphi(\alpha), \varphi(\beta)$ also.

Working a bit more, we deduce

FACT 2. If α, β fill a $\begin{matrix} 1HT \\ 4HS \end{matrix}$ have minimal intersection number then the same holds for $\varphi(\alpha), \varphi(\beta)$.

So here we have deduced that certain configurations of curves are preserved by φ eg.

MAXIMAL CHAIN $\mathcal{M}$

From this we deduce the existence of a homeomorphism $h : S \to S$ st $h|_{\mathcal{M}} = \varphi|_{\mathcal{M}}$. Now have to check that h ~~does~~ agrees with φ on the rest of curves, but you do

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PART 2. INFINITE-TYPE SURFACES.

Now S is connected orientable but $\pi_1(S)$ not f.g.

Examples

S_g - Cantor.

Have the spaces $\mathrm{Ends}(S)$, $\mathrm{Ends}_\infty(S)$ of ends & non-planar ends of S.

Theorem (Kerékjártó, Richards)

① $\mathrm{Ends}(S)$ is a subset of the Cantor set.

② Every pair $X \subset Y \subset$ Cantor is realized as the spaces of (non-planar) ends of some surface.

③ The triple $(\text{genus}, \mathrm{Ends}(S), \mathrm{Ends}_\infty(S))$ determines S up to homeomorphism.

Mapping class group $\mathrm{Mod}(S)$, same definition w/ quotient topology coming from compact-open topology.

$\mathrm{PMod}_{\&}(S)$: the pure mapping class group.

$\mathrm{PMod}_c(S)$: the compactly-supported mapping class group. Generated by Dehn twists. Direct limit of MCGs.

Theorem (Patel - Vlamis '17)

$\mathrm{PMod}(S)$ is $\underset{\text{top.}}{\widehat{\ }}$ generated by $\begin{cases} \text{twists, if } |\mathrm{Ends}_\infty(S)| < 1 \\ \text{twists + handle shifts if } |\mathrm{Ends}_\infty| > 1. \end{cases}$

Finite vs infinite type

- Theorem 1 (BDR, HMV)

$$\mathrm{Aut}(\mathrm{Mod}^{\pm}(S)) = \mathrm{Mod}^{\pm}(S)$$

- Theorem 2 (APV) assume $\mathrm{Ends}_\infty(S) = \mathrm{Ends}(S)$.

$$H^1(\mathrm{PMod}(S), \mathbb{Z}) \cong H_1^{sep}(S, \mathbb{Z}) = \begin{cases} 0 \text{ if } |\mathrm{Ends}(S)| < 1 \\ \mathbb{Z}^{n-1} \quad = n \geq 2 \\ \text{infinite rank if } |\mathrm{Ends}| = \infty \end{cases}$$

Idea of Proof for Theorem 1.

Same idea as Ivanov : prove that twists go to twists.

Here's harder because there is no Nielsen - Thurston classification.

Step 1. Characterize compact support :

$f \in \mathrm{Mod}(S)$ has compact support $\Leftrightarrow$ the conjugacy class of f is countable

($\mathrm{Mod}(S)$ contains ~~at most~~ countably mapping classes w/ compact support)

Step 2. Characterize Dehn twists

Again, the idea boils down, but is not limited to, the center of the centralizer being infinite cyclic.

At this point, we have an induced map $\phi_*: \mathcal{C}(S) \to \mathcal{C}(S)$

Step 3. Prove $\mathrm{Aut}(\mathcal{C}(S)) \cong \mathrm{Mod}^{\pm}(S)$

The idea is to use a direct limit argument & Ivanov's Theorem for finite-type surfaces. Recognize compact subsurfaces in a simplicial way

Step 4. Finish as before.

Corollary. $\mathrm{Mod}^{\pm}(S)$ is Polish.

Idea of proof of Theorem 2. Again, assume $\mathrm{Ends}_\infty(S) = \mathrm{Ends}(S)$.

We want to construct homomorphisms $\mathrm{PMod}(S) \to \mathbb{Z}$.

Remarks. ① $\mathrm{PMod}_c(S)$ is a direct limit of perfect groups, hence perfect.

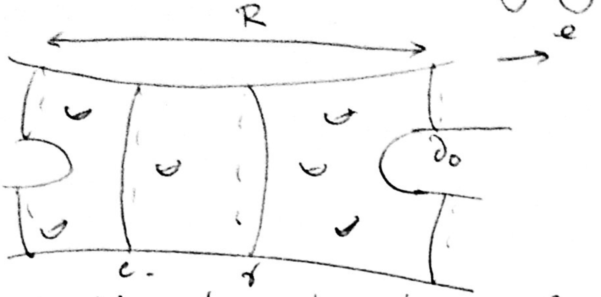
② Dudley. G Polish $\Rightarrow$ $G \to \mathbb{Z}$ continuous. This tells us that $H^1(\overline{\mathrm{PMod}_c}, \mathbb{Z}) = 0$.

So really we are looking at handle-shifts

Let $v \in H_1^{sep}(S, \mathbb{Z})$; assume simple. Note: v determines a partition $\mathrm{Ends}(S) = v^+ \cup v^-$; moreover this partition determines v up to sign.

~~Choose~~ Fix, once and for all, $e \in v^-$

Let $\gamma \in v$. We are going to construct $\phi_\gamma : \mathcal{C}_v(S) \longrightarrow \mathbb{Z}$.



Let $c \in \mathcal{C}_v(S)$. Set

$$\phi_\gamma(c) = g_R(c) - g_R(\gamma).$$

How does ϕ_γ change if we choose another rep of v?

FACT 1. $\phi_\gamma - \phi_\delta = \phi_\gamma(\delta) \in \mathbb{Z}$.

Now observe that $\mathrm{PMod}(S) \curvearrowright \mathcal{C}_v(S)$. We have:

FACT 2. $\forall f \in \mathrm{PMod}(S)$, $\phi_\gamma \circ f - \phi_\gamma = \phi_\gamma(f(\gamma)) \in \mathbb{Z}$.

Theorem (APV) The assignment

$$H_1^{sep}(S,\mathbb{Z}) \xrightarrow{\varphi} H^1(\mathrm{PMod}(S), \mathbb{Z})$$

given by
$$\varphi(v) : \mathrm{PMod}(S) \longrightarrow \mathbb{Z}$$
$$f \longmapsto \phi_\gamma \circ f - \phi_\gamma$$

is a well-defined homomorphism. Moreover:

(1) $\varphi(v)(\mathrm{PMod}_c(S)) = 0$.

(2) If h is a handle-shift then $(\varphi(v)(h) \neq 0 \Leftrightarrow h \text{ cuts } v)$

Remark. We have described the construction for v simple only. For the rest, extend by linearity.